Seat No. : _______________

SK-127

September-2020

B.Sc., Sem.-VI

CC-309 : Mathematics
(Analysis-III)

Time : 2 Hours] **[Max. Marks : 50**

Instructions : (i) Attempt any **THREE** questions in Section-I.

(ii) Section-II is a compulsory section of short questions.

(iii) Notations are usual everywhere.

(iv) The right hand side figures indicate marks of the sub question.

SECTION – I

Attempt any **THREE** of the following questions :

1. (A) Let X be a metric space. Prove that an open sphere is an open set. **7**

(B) Let X be a metric space with metric d. Show that d_1 defined by

$d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$ is also a metric on X. **7**

2. (A) Let X be a metric space. A subset F of X is closed if and only if its complement F' is open. **7**

(B) Let X be a non-empty set, and let d be a real function of ordered pairs of elements of X which satisfies the following two conditions. **7**

$d(x, y) = 0 \Leftrightarrow x = y$, and $d(x, y) \le d(x, z) + d(y, z)$. Show that d is a metric on X.

3. (A) Prove that Compact subsets of metric spaces are closed. **7**

(B) A subset E of the real line R^1 is connected if and only if it has the following

Property : If $x \in E$, $y \in E$ and $x < z < y$, then $z \in E$. **7**

4. (A) Closed subsets of compact sets are compact. **7**

(B) A mapping f of a metric space X into a metric space Y is continuous on X if and only if f^{-1} (V) is open in X for every open set V in Y. **7**

5. (A) State and prove Weierstrass M-test. Show that $f_n(x) = n^2x^n (i - x)$; $x \in [0, 1]$ converges pointwise to a function which is continuous on [0, 1]. **7**

(B) Let (f_n) be a sequence of functions in R [a, b] converging uniformly to f. Then $f \in R[a, b]$ and $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$. **7**

6. (A) Let (f_n) be a sequence of continuous function on $E \subset C$ converges uniformly to f on E, then prove that f is continuous on E. **7**

(B) Let f_n satisfy

(1) $f_n \in D[a, b]$.

(2) $(f_n(x_0))$ converges for $x_0 \in D[a, b]$.

(3) f'_n converges uniformly on [a, b], then prove that f_n converges uniformly on [a, b] to a function f. **7**

7. (A) State and prove Abel's limit theorem. **7**

(B) Show that for $-1 \leq x \leq 1$, $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots.$ Hence evaluate log2. **7**

8. (A) For every $x \in R$ and $n > 0$, prove that **7**

$$\sum_{k=0}^{n} (nx - k)^2 \binom{n}{k} x^k (1 - x)^{n-k} = nx(1 - x) \leq n/4$$

(B) State and prove Weierstrass Approximation theorem. **7**

SECTION – II

9. Attempt any **FOUR** of the followings in short : **8**

(1) Prove that X and ϕ are an open set.

(2) Define : Metric Space.

(3) If F is closed and K is compact, then prove that $F \cap K$ is compact.

(4) Define : Connected set.

(5) Define Uniform convergence.

(6) Prove by Taylor's series $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$