Seat No. : _______________

JF-113

June-2022

M.Sc., Sem.-II

408 : Mathematics

(Real Analysis)

Time : 2 Hours] **[Max. Marks : 50**

Instructions : (1) Each of the questions in **Section-I** carry equal **14** marks.

(2) Attempt any **three** questions from **Section – I**.

(3) Questions in **Section – II** are **COMPULSORY**.

Section – I

1. (A) Define convergence in measure. If the sequence (f_n) of functions converges in measure to two functions f and g, on a bounded set E, prove that f = g almost everywhere on E. **7**

   (B) Let $f_n(x) = e^{-nx^2}$. Show that (f_n) converges to the zero function uniformly on the interval [1, 2]. **7**

2. (A) Let f be a measurable function on a bounded set E. Prove that for any given $\in > 0$, there exists a bounded measurable function g on E such that $mE(f \neq g) < \in$. **7**

   (B) Let $f_n(x) = \begin{cases} 1, & \text{if } x \in [0, 1/n] \\ 0, & \text{if } x \in [1/n, 1] \end{cases}$

   Show that the sequence (f_n) converges in measure to the zero function on [0, 1]. i.e. $f_n(x) \Rightarrow 0$. **7**

3. (A) Let $f_n(x) = \begin{cases} n^2, & \text{if } x \in [0, 1/n] \\ 0, & \text{if } x \in [1/n, 1] \end{cases}$

   Prove that (f_n) converges to the zero function everywhere on [0, 1] but (f_n) does <u>not</u> converge to the zero function in the mean of order p. **7**

   (B) If (f_n) converges in the mean of order p, prove that (f_n) converges to f in measure. **7**

4. (A) Define the space $L^p[0, 1]$ for $p \geq 1$. Prove that $L^p[0, 1]$ is a vector space. **7**

(B) Give an example of a function $f \in L^4[0, 1]$ such that $f \notin L^1[0, 1]$. Give full details. **7**

5. (A) Define a derived number of f at the point x_0. Let $f : [a, b] \to \mathbb{R}$ and $x_0 \in [a, b]$. Prove that f has at least one derived number at x_0. **7**

(B) Define the Cantor function θ carefully. What is the value of the function θ at $\frac{1}{2}$ and at $\frac{1}{6}$? **7**

6. (A) Define absolutely continuous function. Prove that every absolutely continuous function on [a, b] is of bounded variation. **7**

(B) Give an example of a continuous function that is not of bounded variation. **7**

7. (A) What do we mean by an indefinite integral of $f \in L^1[a, b]$?

Prove that the indefinite integral of $f \in L^1[a, b]$ is absolutely continuous. **7**

(B) Define Fourier series of $f \in L^1[-\pi, \pi]$. What is Riemann-Lebesgue lemma ? (Don't prove it) **7**

8. (A) Suppose $f \in L^1[-\pi, \pi]$ and $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$. Show that if f is even then $b_1 = b_2 = ... = 0$. **7**

(B) Define Fourier series of $f \in L^1[-\pi, \pi]$. Show that there is no $f \in L^1[-\pi, \pi]$ whose Fourier coefficients are : $a_k = 0$ for k = 0, 1, 2, 3 ... and $b_k = 1$ for k = 1, 2, 3, ... **7**

Section – II

9. Attempt all : **8**

(1) If $| f(x) - x | < \frac{1}{4}$ for all $x \in [0, 1]$, which of the following functions can be $f(x)$?

(A) x^2 (B) x^3

(C) x^4 (D) $\frac{7x}{8}$

(2) Which of the following subsets of $L^2[0, 1]$ is <u>not</u> dense in $L^2[0, 1]$?

(A) $C[0, 1]$

(B) $C^1[0, 1]$

(C) $P[0, 1]$

(D) The set of all constant functions defined on [0, 1]

(3) The L^2 norm of $f(t) = t$ defined on $[-\pi, \pi]$ is ______.

(A) $\frac{2\pi^3}{3}$ (B) $\sqrt{\frac{2\pi^3}{3}}$

(C) $\frac{2\pi^2}{3}$ (D) $\sqrt{\frac{3\pi^3}{4}}$

(4) Which of the following sequences do not belong to the space l^2 ?

(A) $x_n = 0$ for each n

(B) $x_n = 1$ when n is even; $x_n = 0$ if n is odd

(C) $x_n = \frac{1}{\sqrt{n}}$ for each n

(D) $x_n = \frac{1}{n}$ for each n

(5) What is the total variation $V_a^b(f)$ of $f(x) = x^2$ on $[-1, 1]$?

(A) 0 (B) 1

(C) 2 (D) 4

(6) The derivative of the Cantor function θ is equal to 0 on a set of measure _____.

(A) 1 (B) 1/2

(C) 2 (D) 0

(7) $\int_{-\pi}^{\pi} x^5 \cos x \, dx =$ ______

(A) 0 (B) 1

(C) $\sqrt{2}$ (D) $\sqrt{3}$

(8) Let $p(x)$ and $q(x)$ be polynomials such that $p(n) = q(n)$ for all $n \in \mathbb{N}$. Then______

(A) $p(x) = 0$ and $q(x) = 0$, for all x

(B) $p(x) = q(x)$, for all x

(C) $p(x) = q(x)^2$, for all x

(D) None of these