Seat No. : _______________

JE-104

June-2022

M.Sc., Sem.-II

407 : Mathematics

(Metric Spaces)

Time : 2 Hours] **[Max. Marks : 50**

Instructions : 1. All the questions in **Section-I** carry equal marks.

2. Attempt any **Three** questions in **Section-I.**

3. Questions in **Section-II** are **COMPULSORY.**

Section – I

1. (A) Let d_1 be a metric on X. Define $d_2(x, y) = \min\{1, d_1(x, y)\}$ for all $x, y \in X$. Show that d_2 is a metric on X. Is d_2 bounded metric ? Justify your answer. **7**

(B) Let $U = \{(x, y) \in \mathbb{R}^2 : x \notin \mathbb{Z}, y \notin \mathbb{Z}\}$. Is U open in $\mathbb{R}^2$? Justify your answer. **7**

2. (A) Prove that a nonempty open set in $\mathbb{R}$ is the union of countable family of pairwise disjoint open intervals. **7**

(B) Show that unit circle in $\mathbb{R}^2$ is closed. **7**

3. (A) Let X be a metric space and $E \subset X$. Prove that a point x is a limit point of E if and only if there exists a sequence (x_n) in E such that $x_n \to x$. **7**

(B) Define closure of a set. Show that $\overline{A}$ is the smallest closed set containing A. Find the closure of $(-1, 0) \cup \mathbb{N}$ in $\mathbb{R}$. **7**

4. (A) State and prove Bolzano Weierstrass theorem. **7**

(B) Show that $\mathbb{R}$ is complete. **7**

5. (A) Let X, Y be metric spaces. Show that a map $f : X \to Y$ is continuous if and only if for every open set $V \subset Y$, its inverse image $f^{-1}(V)$ is open in X. **7**

(B) Let X, Y be metric spaces. Let $f, g : X \to Y$ be continuous.

Is the set $E = \{x \in X : f(x) \neq g(x)\}$ open in X ? Justify your answer. **7**

6. (A) State Gluing Lemma (Do not prove.). Let $f, g : [0, 1] \to X$ be continuous. Assume that $f(1) = g(0)$. Define

$$h(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Show that h is continuous. **7**

(B) Define uniformly continuous function. **7**

Discuss the uniform continuity of the following functions :

(a) $f : [1, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$.

(b) $g : (0, 1) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$.

7. (A) Prove that any compact subset of a metric space is closed and bounded. **7**

(B) Is $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ compact in $\mathbb{R}$? Justify your answer. **7**

8. (A) Show that the continuous image of a connected space is connected. **7**

(B) Show that $\mathbb{R}$ is connected. Is $\mathbb{R}^*$ the set of all nonzero real numbers connected ? Justify your answer. **7**

Section-II

9. Attempt all of the following : **8**

(1) If d is a discrete metric on $\mathbb{R}$, then d(1000, 2022) = _____.

(A) 1 (B) 1022

(C) 0 (D) ∞

(2) Which of the following sets is both open and closed in $\mathbb{Q}$?

(A) $\{x \in \mathbb{Q} : -5 < x < 5\}$

(B) $\{x \in \mathbb{Q} : -\sqrt{5} < x < \sqrt{5}\}$

(C) $\{x \in \mathbb{Q} : -\sqrt{5} < x < 5\}$

(D) $\{x \in \mathbb{Q} : -5 < x < \sqrt{5}\}$

(3) Which of the following subsets is dense in $\mathbb{R}$?

(A) $\{x \in \mathbb{R}/x \geq 0\}$ (B) $\mathbb{Z}$

(C) $\mathbb{N}$ (D) $\mathbb{R}$

(4) Consider A = $\{(x, y) : x^2 + y^2 < 1, y > 0\}$ U $\{(x, y) : -1 \leq x \leq 1, y = 0\} \subset \mathbb{R}^2$. What is the boundary ∂A of A in $\mathbb{R}^2$?

(A) $\{(x, y) : x^2 + y^2 = 1\} \cup \{(x, y) : -1 \leq x \leq 1, y = 0\}$

(B) $\{(x, y) : x^2 + y^2 = 1, y < 0\} \cup \{(x, y) : -1 \leq x \leq 1, y = 0\}$

(C) $\{(x, y) : x^2 + y^2 = 1, y > 0\} \cup \{(x, y) : -1 \leq x \leq 1, y = 0\}$

(D) $\{(x, y) : x^2 + y^2 = 1\}$

(5) What is the distance d(A, B) between the sets A = $\mathbb{Q}$ and B = (–1, 1) ?

(A) 0 (B) $\frac{1}{2}$

(C) 1 (D) 2

(6) Which of the following statements is not true ?

(A) Any two open balls in $\mathbb{R}^n$ are homeomorphic.

(B) Circle and ellipse in $\mathbb{R}^2$ are homeomorphic

(C) Circle and hyperbola in $\mathbb{R}^2$ are homeomorphic

(D) Closed intervals [0,1] and [0, 3] in $\mathbb{R}$ are homeomorphic.

(7) Which of the following subsets of $\mathbb{R}$ is compact ?

(A) $\mathbb{Z}$ (B) $\mathbb{Q}$

(C) $\{x \in \mathbb{N} : 0 < x < 100\}$ (D) $(0, 1)$

(8) Which of the following subsets of $\mathbb{R}^2$ is connected ?

(A) $\{(x, y) : xy \neq 0\}$

(B) $\{(x, y) : x^2 + y^2 = 1\}$

(C) $\{(x, y) : x \in \mathbb{Q}, y \notin \mathbb{Q}\}$

(D) $\{(x, y) : x^2 - y^2 = 1\}$