Seat No. : _______________

AH-131

April-2022

B.Sc., Sem.-VI

307 : Mathematics

(Abstract Algebra – II)

Time : 2 Hours] **[Max. Marks : 50**

Instructions : (1) Attempt any **three** questions in **Section – I**.
(2) **Section – II** is a compulsory section of short questions.
(3) Notations are usual everywhere.
(4) The right hand side figures indicate marks of the sub. question.

SECTION – I

Attempt any **three** of the following questions :

1. (a) Prove that the set $Z[i] = \{a + ib/a, b \in Z\}$ forms a commutative ring under usual addition and multiplication of complex numbers. **7**

(b) If R is ring satisfying $(a + b)^2 = a^2 + 2ab + b^2$ for all $a, b \in R$, then prove that the ring R is a commutative ring. **7**

2. (a) Define an integral domain and prove that every field is an integral domain. Also justify your answer whether the converse is true or false. **7**

(b) If R is a ring satisfying $a^2 = a$, $\forall a \in R$, then prove that R is a commutative ring. **7**

3. (a) Define a subring of a ring R.

If U is a non-empty subset of a ring R satisfying

(i) $a - b \in U$ for all $a, b \in U$ and

(ii) $a \cdot b \in U$ for all $a, b \in U$ then prove that U is a subring of R. **7**

(b) Show that the set $U = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \Big/ a, b, c \in Z \right\}$ of triangular matrices is a subring of the ring $(M_2(Z), +, \bullet)$. **7**

4. (a) If R is a commutative ring with unity which has no proper ideal, then prove that R is a field. **7**

(b) Define a ring Homomorphism. If $\Phi : (R, +, \bullet) \to (R', \oplus, \odot)$ is a ring homomorphism and I is an ideal of R, then prove that $\Phi(I)$ is an ideal of $\Phi(R)$. **7**

5. (a) For non-zero polynomials $f, g \in D[x]$ prove that $[f \cdot g] = [f] + [g]$. **7**

(b) Obtain the quotient $q(x)$ and the remainder $r(x)$ on dividing $f(x) = x^4 + 2x^3 - x^2 + 3$ by $g(x) = x^2 - x + 1$ in $Z_5[x]$. **7**

6. (a) Suppose $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \in Z[x]$ and suppose $\frac{p}{q}$ in the simplest form (i.e. $(p, q) = 1$) is a solution of the equation $f(x) = 0$. Then prove that $p|a_0$ and $q|a_n$. **7**

(b) State the Eisenstein's criterion and prove that whenever p is a prime, then the polynomial $f(x) = x^{p-1} + x^{p-2} + \ldots + x + 1$ is irreducible over Q. **7**

7. (a) Show that $Q[i] = \left\{ \frac{a + bi}{a, b \in Q} \right\}$ is a subfield of the field C of complex numbers. **7**

(b) Show that the polynomial $f(x) = x^3 + x^2 + 1$ is irreducible over Z_2 and $\frac{Z_2[x]}{< f(x) >}$ is a field with eight elements. **7**

8. (a) If I is a maximal ideal of a commutative ring R with unity, then prove that R/I is a field. **7**

(b) Show that $I = \langle x^3 - 3x - 1 \rangle$ is a maximal ideal in $Z_3[x]$. **7**

SECTION – II

9. Attempt any **four** of the followings **in short** : **8**

(a) Define the terms : (i) A ring (ii) A commutative ring.

(b) Give an example of an infinite non-commutative ring and a finite commutative ring.

(c) Give an example of a left ideal of a ring which is not an ideal of the ring.

(d) Define a polynomial in integral domain D and the degree of a non-zero polynomial in D.

(e) Define a primitive polynomial and a reducible polynomial.

(f) Define maximal and prime ideals of a commutative ring R with unity.
