XW-112

April-2013

M.Sc. (Sem. II)
407 - MATHEMATICS
(Differential Geometry – I)

Time : 3 Hours] **[Max. Marks : 70**

1. (A) Let $\bar{r}(t)$ be the logarithmic spiral

$$\bar{r}(t) = (e^t \cos(t), e^t \sin(t)).$$

Find the arc-length of $\bar{r}$ starting at $\bar{r}(0) = (1, 0)$. **7**

OR

Calculate the arc-length of the catenary

$\bar{r}(t) = (t, \cosh(t))$ starting at the point $(0, 1)$.

(B) Answer briefly any **two** : **4**

(i) Is the curve $\bar{r}$ given below a unit speed curve ?

$$\bar{r}(t) = \left(\frac{5}{13}\sin(t), \frac{12}{13}\sin(t), 1 - \cos(t)\right).$$

(ii) Show that the length of the twisted cubic

$\bar{r}(t) = (t, t^2, t^3)$, between the points $(0, 0, 0)$ and $(1, 1, 1)$ is less than 4.

(iii) Sketch the curve $\bar{r}(t) = (t, \log(t))$, $0 < t < \infty$. Sketch the curve $\bar{\delta}(u) = (e^u, u)$, $-\infty < u < \infty$. How are these parametrited curves related ?

(C) Answer very briefly all **three** : **3**

(i) Sketch the curve $\bar{r}(t) = (-\cos(t), \sin(t))$, $0 < t < \frac{\pi}{2}$.

(ii) Parametrite the line passing through the points $(1, 2, 3)$ and $(2, 3, 4)$.

(iii) Is the curve $\bar{r}$ given below regular ?

$$\bar{r}(t) = (t^2, t^3), -\infty < t < \infty.$$

2. (A) Compute k, τ, $\bar{t}$, $\bar{n}$ and $\bar{b}$ for the curve **7**

$$\bar{r}(t) = \left(\frac{t}{\sqrt{2}}, \frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}\right).$$

OR

Compute k, τ, $\bar{t}$, $\bar{n}$ and $\bar{b}$ for the curve

$$\bar{r}(t) = \left(\frac{20}{29}\cos(t), \frac{-21}{29}\cos(t), 1 - \sin(t)\right)$$

(B) Answer briefly any **two** : **4**

(i) Is the curve given below a planar curve ?

$$\bar{r}(t) = \left(\frac{t^2-1}{t}, t+1, \frac{1+t}{t}\right).$$

(ii) Give an example of a unit speed plane curve whose signed curvature is –1 at every point.

(iii) Suppose $\bar{r}(s)$ is a unit speed curve and $\bar{\alpha}(s)$ is the curve given by $\bar{\alpha}(s) = 2\bar{r}s$. How are the curvatures of $\bar{\alpha}$ and $\bar{\gamma}$ related at points with the same value of the parameter s ?

(C) Answer very briefly all **three** : **3**

(i) Write down (without proof) a formula for the torsion τ of a regular curve $\bar{r}(t)$ in $\mathbb{R}^3$.

(ii) Write down (without proof) the Frenet-Serret equations.

(iii) Give a parametrization for a circular helix.

3. (A) Show that the level surface

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{5^2} = 1 \text{ is a smooth surface.}$$

Roughly sketch this surface. **7**

OR

Find the equation of the tangent plane to the surface

$$\bar{\sigma}(\theta, \phi) = ([3 + \cos(\theta)] \cos(\phi), [3 + \cos(\theta)] \sin(\phi), \sin(\theta))$$

at the point (–3, 0, 1).

(B) Answer briefly any **two** : **4**

(i) Show that an open disc in the xy-plane is a smooth surface.

(ii) Give a parametrization $\bar{\sigma}(\theta, \phi)$ for the unit sphere in terms of the latitude θ and the longitude ϕ.

(iii) Show that the unit sphere cannot be covered by a single surface patch.

(C) Answer very briefly all **three** : **3**

(i) Give a parametrization for the plane

$$S = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 3z - 5 = 0\}.$$

(ii) Define the tangent space at a point P of a surface S.

(iii) Define the standard unit normal of the surface patch $\bar{\sigma}$ at the point $P = \bar{\sigma}(u_0, v_0)$.

4. (A) Identify the quadric given by

$$x^2 - 2y^2 + 6x + 4y + z = 5,$$

by first transforming its equation. **7**

OR

Define a triply orthogonal system of surfaces.

Show that the following form a triply orthogonal system :

the planes parallel to the xy-plane,
the planes parallel to the yz-plane,
the planes parallel to the xz-plane.

(B) Answer briefly any **two** : **4**

(i) Define a generalized cylinder and give its parametrization (without proof).

(ii) Define a generalised cone and give its parametrization (without proof).

(iii) Define a ruled surface and give its parametrization(without proof).

(C) Answer very briefly all **three** : **3**

(i) Give an equation of a hyperboloid of one sheet.

(ii) Give an example of a one-parameter family of spheres.

(iii) Give a parametrization $\bar{\sigma}(u, v)$ for the surface of revolution obtained by rotating the profile curve $\bar{r}(u) = (f(u), 0, g(u))$ around the z-axis. (Do not prove).

5. (A) Calculate the first fundamental forms of the following surfaces : **7**

(i) $\bar{\sigma}(u, v) = (u - v, u + v, u^2 + v^2)$,

(ii) $\bar{\sigma}(u, v) = (\cosh(u), \sinh(u), v)$.

OR

Define an isometry of two surfaces S_1 and S_2.

Show that $S_1 = \{(x, y, 0) \in \mathbb{R}^3 : 0 < \times < 2\pi\}$

and $S_2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, x \neq 1\}$ are isometric.

(B) Answer briefly any **two** : **4**

(i) Define a conformal map between two surfaces S_1 and S_2.

(ii) Determine the area of the part of the paraboloid $z = x^2 + y^2$ with $z \leq 1$.

(iii) Give (without proof) a necessary and sufficient condition for $f : S_1 \to S_2$ to be an isometry, in terms of the first fundamental form.

(C) Answer very briefly all **three** : **3**

(i) Define an equiareal map between two surfaces S_1 and S_2.

(ii) State (without proof) Archimedes theorem.

(iii) Express $\left\| \bar{\sigma}_u \times \bar{\sigma}_v \right\|$ in terms of E, F, and G. (without proof).
