Seat No. : ______________

AD-108

April-2016

B.Sc., Sem.-VI

CC-309 : Mathematics
(Analysis – III)

Time : 3 Hours] **[Max. Marks : 70**

Instructions : (1) **All** questions are compulsory.

(2) Write the question number in your answer sheet as shown in the question paper.

(3) Figures to the right indicate marks of the question.

1. (a) Prove : In any metric space X, each open sphere is an open set. **7**

OR

Prove that every Cauchy sequence is bounded.

(b) If $x = (x_1, x_2)$, $y = (y_1, y_2) \in R^2$ and if the mapping $d : R^2 \times R^2 \rightarrow R$ is defined by $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ then prove that d is a metric for R. **7**

OR

Prove : B dry A is closed set, for G subset A of a metric space X.

2. (a) Prove that the continuous image of a compact metric space is compact. **7**

OR

Let F be a bounded and closed set in R. Then prove that every open covering of F has a finite sub covering.

(b) Prove that the set A = (0, 2) has no separation. **7**

OR

Prove that the function f : (0, 1) → R defined by $f(x) = \frac{1}{x}$ is not uniformly continuous.

3. (a) Let (f_n) be a sequence of continuous function on [a, b] and suppose that $f_n \to f$ uniformly on [a, b] then prove $\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n \to \infty} f_n(x)\, dx$. **7**

OR

Suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in E$ and let $M_n = \text{Sup}_{x\in E} | f_n(x) - f(x) |$. Then prove that $f_n \to f$ uniformly on E if and only if $\lim_{n\to\infty} M_n = 0$.

(b) Let $f_n(x) = |x|^{1+\frac{1}{n}}$ for $x \in [-1, 1]$ then show that

(i) $f_n \in D[-1, 1]$

(ii) $f_n(x) \to f(x) = |x|$ uniformly on [–1, 1]

(iii) f is not differentiable. **7**

OR

Show that the sequence (f_n) ; where $f_n(x) = \frac{x}{1 + n^2x^2}$ $\forall x \in [-1, 1]$ converges uniformly and the limit function is differentiable ; but the relation $f'(x) = \lim_{n\to\infty} f'_n(x)$ does not hold good for all $|x| \leq 1$.

4. (a) If $f(x) = \sum a_n x^n$ be power series with radius of convergence 1, if the series converges at 1, then prove that $\lim_{x \to 1} f(x) = f(1)$. **7**

OR

Prove that (i) $\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$ and

(ii) $\sum_{k=0}^{n} (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x) \le \frac{n}{4}$

For every $x \in R$ and $n \ge 0$.

(b) For $-1 \le x \le 1$ prove that $(\tan^{-1}x)^2 = 2\left[\frac{x^2}{2} - \left(1 + \frac{1}{3}\right)\frac{x^4}{4} + \left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{x^6}{6} - \cdots\right]$ and

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ **7**

OR

Show that the function $f(x) = \begin{cases} e^{1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$ has derivatives of all orders at all $x \neq 0$ but does not have a Taylor's theorem.

5. Give the answer in brief : (any **seven**) **14**

(1) Define interior point in metric space.

(2) Find the limit points of the set of all rational numbers, Q and the set of real numbers R.

(3) Show that (0, 1) is not compact.

(4) Say true or false : If the series Σa_k converges absolutely then the series $\Sigma a_k \cos kx$ is not uniformly convergent on R.

(5) Define connected metric space.

(6) State Intermediate value theorem.

(7) State Taylor's theorem.

(8) Define uniform convergence.

(9) If R is the radius of convergent of the series $\Sigma\, na_n x^{n-1}$ then what is the radius of convergent of the series $\Sigma\, n(n-1)a_n x^{n-2}$?