Seat No. : _______________

AC-122

April-2016

M.Sc., Sem.-IV

508 : Mathematics
(Fourier Analysis)

Time : 3 Hours] **[Max. Marks : 70**

1. (A) Attempt any **one**. **(7)**

   (1) State and prove the uniqueness theorem for the real-valued continuous and L^1 functions.

   (2) If $f \in L^\infty$ then show that

   $$\lim_{p \to \infty} \|f\|_p = \|f\|_{L^\infty}.$$

   (B) Attempt any **two**. **(4)**

   (1) Does there exist a non-constant function $f \in L^1$ such that $\hat{f}(mn) = m\hat{f}(n)$ for all non-zero integers m and n ?

   (2) If f is absolutely continuous, then show that $\widehat{Df}\,(n) — in\hat{f}(n)$.

   (3) If $f \in L^1$, then show that

   $$\hat{\bar{f}}(n) = \overline{\hat{f}(-n)}.$$

   (C) Answer in brief. **(3)**

   (1) If $f(x) = 3e^{i2x} - 2ie^{-ix} + 5$ and $g(x) = f(x - 2)$, then what is $\hat{g}(2)$?

   (2) Show that the Fourier transform map $T : L^1 \to l_\infty(Z)$ is continuous.

   (3) Give an example of an unbounded function f in L^∞.

2. (A) Attempt any **one**. **(7)**

   (1) Let $\{K_n\}$ be an approximate identity and $1 \le p < \infty$. Then show that

   $$\lim_{n \to \infty} \|K_n * f - f\|_p = 0, \forall f \in L^p.$$

   (2) If γ is a non-trivial complex continuous algebra homomorphisms between L^1 and the space of complex numbers C, then show that there exists a unique positive integer N such that $\gamma(f) = \hat{f}\,(N)$, for every $f \in L^1$.

(B) Attempt any **two**. **(4)**

(1) Prove that

$$T_a(f * g) = T_a f * g = f * T_a g.$$

(2) Does there exist two distinct elements in L^1 which are idempotent but whose sum is not an idempotent element ? Justify your answer.

(3) Does there exist f, g $\in L^1$ which are not trigonometric polynomials but for which $f * g \equiv 0$? Justify your answer.

(C) Answer in brief **(3)**

(1) Show that convolution is commutative.

(2) If $f(x) = 3e^{i2x}$, then what is $f * f * f$?

(3) True or False : L^1 has zero divisors with respect to convolution.

3. (A) Attempt any **one**. **(7)**

(1) If $f \in L^1$, then prove that

$$\int_a^b f(x)\,dx = \hat{f}(0)(b-a) + \sum_{n \neq 0} \hat{f}(n)\frac{e^{inb} - e^{ina}}{in}$$

(2) Define Fejer kernel $F_N(x)$ and show that the sequence $\{F_N\}$ is an approximate identity for convolution in L^1.

(B) Attempt any **two**. **(4)**

(1) State (only) Fejer's theorem.

(2) Suppose f, g $\in L^1$ and Fourier series of g converges a.e. essentially boundedly. Then show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,g(x) = \sum_{n \in Z} \hat{f}(n)\,\hat{g}(-n).$$

(3) If $f \in L^1$ is such that $\hat{f}(n) = 0$ for all n, then show that $\sigma_N f(x) = 0$ for all N and all x.

(C) Answer in brief. **(3)**

(1) Show that $S_N f = f * D_N$ where $S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$.

(2) For the series $\sum c_n$, state (only) any one condition under which cesaro summability implies summability.

(3) True or False: If $a_n = \int_{-\pi}^{\pi} D_n(x)dx$ and $b_n = \frac{a_{n+1}}{n}$, then (b_n) is a bounded sequence.

4. (A) Attempt any **one**. **(7)**

(1) If $a_n \downarrow 0$ and $\sum a_n \sin nx$ converges uniformly then show that $na_n \to 0$ as $n \to \infty$.

(2) If (a_n) is quasi-convex and bounded, then show that the sequence $(n\Delta a_n)$ is bounded. Also show that if (a_n) is quasi-convex and convergent then the sequence $(n\Delta a_n)$ is convergent.

(B) Attempt any **two**. **(4)**

(1) The sine series $\sum_{N=2}^{\infty} \frac{\sin nx}{\log n}$ converges everywhere but it is not a Fourier series. Explain this.

(2) If $a_n \to 0$ and $\sum |\Delta a_n| < \infty$, then show that the cosine series $\sum a_n \cos nx$ converges uniformly in $[-\pi, \pi] - [-\delta, \delta]$.

(3) Prove of disprove: If $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$, then f is continuous.

(C) Answer in brief. **(3)**

(1) Is $a_n = \frac{1}{\log n}$ convex ?

(2) True or False : The Fourier transform map $T : L^1 \to C_0(Z)$ is onto.

(3) True or False: If $a_n \downarrow 0$, then $\sum a_n \cos nx$ converges everywhere.

5. (A) Attempt any **one**. **(7)**

(1) State the Uniform Boundedness theorem and using it show that there exists a function which is continuous at 0 but whose Fourier series diverges at 0.

(2) If $1 \leq p < \infty$, then show that $L^p \subseteq L^1 * L^p$.

(B) Attempt any **two**. **(4)**

(1) If f is of bounded variation then show that $\{n\hat{f}(n)\}$ is a bounded sequence.

(2) If (b_n) is a sequence of non-negative real numbers converging to 0, then show that there exists a sequence (a_n) of non-negative real numbers such that :

(i) $\sum a_n = \infty$,

(ii) $\sum a_n b_n < \infty$ and

(iii) $\sum \frac{a_n}{n} < \infty$.

(3) If $f \in L^1$ then show that $\sum_{n \neq 0} \frac{\hat{f}(n)e^{inx}}{n}$ converges uniformly.

(C) Answer in brief. **(3)**

(1) State (only) Dini's test for convergence of Fourier series.

(2) True or False: $L^2 * L^2 = L^2$.

(3) True or False: If $f \in L^1$ is continuous and of bounded variation everywhere then the Fourier series of f converges uniformly.