Seat No. : _______________

AR-128

May-2016

M.Sc., Sem.-II

408 : Mathematics
(Algebra – I)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any **one**. **7**

(i) If $\varepsilon = \beta_1\beta_2....\beta_r$, where β's are 2-cycles, prove that r is even.

(ii) State and prove the Orbit-Stabilizer Theorem.

(b) Attempt any **two**. **4**

(i) Can a group have more subgroups than it has elements ? Justify.

(ii) Show that for $n \geq 3$, $Z(S_n) = \{\varepsilon\}$

(iii) Let G be the group of non-zero complex numbers under multiplication and let $H = \{x \in G / |x| = 1\}$. Give a geometric description of the cosets of H.

(c) Answer very briefly. **3**

(i) Show that group $(\mathbb{Z}, +)$ is not isomorphic to $(\mathbb{Q}, +)$

(ii) Prove or disprove : The group $\mathbb{Z} \oplus \mathbb{Z}$ is cyclic.

(iii) If $\alpha = (12)(134)(152)$ then α is even or odd ? Justify.

2. (a) Attempt any **one**. **7**

(i) Let G be a finite Abelian group and let p be a prime that divides the order of G, prove that G has an element of order p.

(ii) Let G and H be finite cyclic groups, prove that $G \oplus H$ is cyclic iff $|G|$ and $|H|$ are relatively prime.

(b) Attempt any **two**. **4**

(i) If G = U(32) and H = {1,31}. Find the isomorphism class of the factor group G/H.

(ii) In the group $\mathbb{Z}$, let H = < 5 > and K = <7>. Prove that $\mathbb{Z}$ = HK. Does $Z = H \times K$? Justify.

(iii) Determine the last two digits of 23^{23}.

(c) Answer very briefly. **3**

(i) What is the largest order of any element in $\mathbb{Z}_{30} + \mathbb{Z}_{20}$? Explain.

(ii) If $|G| = pq$, where p and q are not necessarily distinct primes, show that $|Z(G)| = 1$ or pq.

(iii) What is the order of element 5+ <6> in the factor group $\mathbb{Z}_{18}/ <6>$?

3. (a) Attempt any **one**. **7**

(i) State and prove N/C theorem.

(ii) State and prove the First Isomorphism Theorem.

(b) Attempt any **two**. **4**

(i) If ϕ is a homomorphism from Z_{30} onto a group of order 5, determine the Kernel of ϕ.

(ii) Find all Abelian groups (up to isomorphism) of order 360.

(iii) Let G be a group. If $H = \{g^2/g \in G\}$ is a subgroup of G, prove that it is a normal subgroup of G.

(c) Answer very briefly. **3**

(i) Prove that a factor group of an Abelian group is Abelian.

(ii) Define normal subgroup. Show that S_n has a proper, nontrivial normal subgroup.

(iii) Show that group $G \oplus \overline{G}$ is isomorphic to $\overline{G} \oplus G$.

4. (a) Attempt any **one**. 7

(i) State and prove Sylow's first theorem.

(ii) State and prove Sylow's second theorem.

(b) Attempt any **two**. **4**

(i) Show that a noncyclic group of order 21 must have 14 elements of order 3.

(ii) How many Sylow 5-subgroups of S_5 are there ? Exhibit two.

(iii) If G has a unique Sylow p-subgroup H, show that H is normal in G.

(c) Answer very briefly. **3**

(i) Suppose $|G| = 168$. If G has more than one Sylow 7-subgroup, exactly how many does it have ? Explain.

(ii) Let $a \in G$. Define the conjugacy class cl(a). Show that cl(a) = {a} iff $a \in Z(G)$.

(iii) Define the terms : (i) The normalizer N(H) (ii) The centralizer C(H).

5. (a) Attempt any **one**. 7

(i) Define simple group. Prove that A_5 is simple.

(ii) State and prove Generalized Cayley Theorem.

(b) Attempt any **two**. **4**

(i) Does there exist a simple group of order 112 ? Justify.

(ii) Show that A_5 cannot contain a subgroup of order 30, 20 or 15.

(iii) How many non-equivalent ways are there to string three black and three white beads on a necklace of six beads under the dihedral group D_6 ? Explain briefly.

(c) Answer very briefly. **3**

(i) If $|G| = p^n$, where p is prime and $n \geq 2$, show that G cannot be simple.

(ii) Define (a) orbit (i) (b) fix (ϕ).

(iii) State (without proof) : (i) Embedding theorem (ii) Burnside's theorem

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