Seat No. : _______________

AA-112

April-2016

M.Sc., Sem.-IV

507 : Mathematics
(Functional Analysis – II)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any **one** : **7**

(i) Let H be a Hilbert space, and let f be an arbitrary functional in H* then prove that there is a unique vector y in H such that $f(x) = (x, y)$ for all x in H.

(ii) If A is a positive operator on H, then prove that I + A is non-singular. Hence prove that I + T * T and I + TT* are non-singular for any T in B(H).

(b) Attempt any **two** : **4**

(i) Let H = $\mathbb{R}^2$, K = $\mathbb{R}$ and A $\in$ BL(H) is given by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that A is positive if and only if $b = c$, $a \geq 0$, $d \geq 0$, $ad \geq b^2$.

(ii) Let H be a Hilbert space and T $\in$ B(H) be non-zero, self-adjoint then prove that T^n is also non-zero, self-adjoint. (Here, n is a positive integer)

(iii) By an example prove that $0 \leq A \leq B$ does not imply $A^2 \leq B^2$.

(c) Answer very briefly. **3**

(i) Prove that $\| f_y \| = \| y \|$ for all y in H.

(ii) If H is a Hilbert space over $\mathbb{R}$, prove that the mapping $y \rightarrow f_y$ form H to H* is linear.

(iii) Define the normal operator. Give an example of a normal operator that is not unitary.

2. (a) Attempt any **one** : **7**

(i) Prove : $T^*T = I \Leftrightarrow (Tx, Ty) = (x, y)$ for all x, y $\Leftrightarrow \| Tx \| = \| x \|$ for all x.

(ii) If P is the projection on M, then prove that

$x \in M \Leftrightarrow Px = x \Leftrightarrow \| Px \| = \| x \|$.

(b) Attempt any **two**. **4**

(i) Show that the unitary operators on H form a group.

(ii) If P and Q are projections on M and N respectively. Under what condition(s) does PQ become a projection ? What is the range of PQ ?

(iii) Give an example of a linear map on R^2 that does not have eigen value.

(c) Answer very briefly : **3**

(i) If $T = \alpha I$, then find the spectrum of T.

(ii) Define the Spectral resolution of T.

(iii) Give an example of a positive operator that is not a projection.

3. (a) Attempt any **one**. **7**

(i) State and prove the spetral theorem in the case of finite dimensional H.

(ii) Prove that two matrices in A_n are similar if and only if they are the matrices of a single operator on H relative to (possibly) different bases.

(b) Attempt any **two** : **4**

(i) If T is non-singular, prove that $k \in \sigma(T) \Leftrightarrow 1/k \in \sigma(T^{-1})$

(ii) If $T \in B(H)$ and N a normal operator. Show that T commutes with N* if T commutes with N.

(iii) If $T^k = 0$ for some positive integer k then prove that $\sigma(T) = \{0\}$.

(c) Answer very briefly. **3**

(i) Can an 11×11 real matrix have the empty spectrum? Justify.

(ii) True or false : The map T on $\mathbb{R}^2$ defined by $T(x, y) = (x - y, x + y)$ is invertible. Justify.

(iii) Find the matrix corresponding to the map $A(x, y) = (y, x)$ on $\mathbb{R}^2$.

4. (a) Attempt any **one**. **7**

(i) State and prove Gelfand-Mazur theorem.

(ii) If X is a Banach space and $A \in BL(X)$, prove that A is invertible if and only if A is bijection.

(b) Attempt any **two** : **4**

(i) Find the spectrum of $A(x) = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$ where $x \in l^p$.

(ii) Characterize the approximate eigenspectrum $\sigma_a(T)$ of T.

(iii) Define the spectral radius $r_\sigma(A)$ of A. Give an example to show that $r_\sigma(A)$ can be strictly less than $\|A\|$.

(c) Answer very briefly : **3**

(i) Is the operator $A(x, y, z) = (0, y, z)$ on $\mathbb{R}^3$ invertible ? Justify.

(ii) Find an operator A such that $\sigma(A) = [0,1]$.

(iii) True or false : Every non-zero projection is invertible. Justify.

5. (a) Attempt any **one**. **7**

(i) Let Y be a Banach space, $F_n \in CL(X, Y)$, $F \in BL(X, Y)$ and $F_n \to F$. Then prove that $F \in CL(X, Y)$.

(ii) Prove that $F \in BL(X, Y)$ is compact if and only if for every bounded sequence (x_n) in X. $(F(x_n))$ has a subsequence which converges in Y.

(b) Attempt any **two**. **4**

(i) Prove that every $m \times n$ matrix defines a compact map.

(ii) Prove that CL(X, Y) is a linear subspace of BL(X, Y).

(iii) If A is a compact operator on X, prove that the eigenspace of A corresponding to a non-zero eigen value of A is finite dimensional.

(c) Answer very briefly : **3**

(i) True or false : $A(x) = \alpha x$ (where α is a non-zero scalar) is a compact operator on l^2. Justify.

(ii) Prove or disprove the linear map T from $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (y, x)$ is compact.

(iii) Can we find a compact operator A such that $\sigma_a(A) = [0.1]$? Justify.
