Seat No. : _______________

NQ-128

December-2015

M.Sc., Sem.-I

402 : Mathematics

(Metric Spaces)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Define an open ball in a metric space (X, d). Define an open set in the metric space (X, d). Show that if U and V are two open subsets of X, then $U \cap V$ is also open. **7**

OR

Define an open ball in a metric space (X, d).

Consider C[0, 1] with the metric given by the sup norm, i.e. for f, g $\in$ C [0, 1],

$d(f, g) = \sup \{|f(x) - g(x)| : x \in [0, 1]\}$.

Let $f(x) = x^2$ and $g(x) = x$. Does f belong to the ball $B\left(g, \frac{1}{3}\right)$?

(b) Answer any **two** : **4**

(i) Find two open sets U, V in $\mathbb{R}^3$ such that $(1, 2, 3) \in U$, $(2, 3, 4) \in V$ and $U \cap V = \phi$.

(ii) Give an example of an unbounded closed subset of $\mathbb{R}^2$ which does not contain the point (0, 1).

(iii) Show that the interval (3, 8) in $\mathbb{R}$ is an open ball.

(c) Answer **all** : **3**

(i) State (without proof) the Hausdorff property.

(ii) Let (Y_1, d_1) and (Y_2, d_2) be metric spaces. Define (without proof) the product metric on $Y_1 \times Y_2$.

(iii) Let X be a nonempty set. List the properties that a function $d : X \times X \longrightarrow \mathbb{R}$ must satisfy for d to be a metric on X.

2. (a) Let (X, d) be a metric space.

Let (x_n) be a sequence in X.

Define : (x_n) converges to $x \in X$. In $\mathbb{R}^2$, consider the sequence whose n^{th} term is $\left(1 + \frac{1}{n}, 2 - \frac{1}{n}\right)$. Find (giving details) the limit of this sequence. **7**

OR

Let X be a metric space. Let E be a subset of X.

Define : $x \in X$ is a limit point of E.

Show that if E is closed, it contains all its limit points.

(b) Answer any **two** : **4**

(i) State (without proof) the Bolzano Weierstrass Theorem.

(ii) Consider the set SL $(2, \mathbb{R})$ of all 2×2 real matrices with determinant 1. Is this set a bounded subset of M $(2, \mathbb{R})$?

(iii) Give an example of an open subset U of $\mathbb{R}$, such that U is dense in $\mathbb{R}$ and $U \neq \mathbb{R}$.

(c) Answer **all** : **3**

(i) Define : (x_n) is a Cauchy sequence in (X, d).

(ii) State (without proof) the Weierstrass Approximation theorem.

(iii) Find a number in the interval $\left(1, \frac{3}{2}\right)$, of the form $n + m\sqrt{2}$, where $n, m \in \mathbb{Z}$.

3. (a) Let (X, d_1) and (Y, d_2) be metric spaces.

Let $f : X \to Y$ be a function.

Define : f is continuous at $x \in X$. Define : f is continuous on X.

Suppose f is continuous at $x \in X$.

Show that given $\Sigma > 0$, there exists a $\delta > 0$, such that if $d_1(x, y) < \delta$ then $d_2(f(x), f(y)) < \Sigma$. **7**

OR

Let (X, d_1) and (Y, d_2) be metric spaces.

Let $f : X \to Y$ be a function.

Define : f is continuous at $x \in X$. Define : f is continuous on X.

Suppose $f : X \to Y$ is continuous. Let V be a closed subset in Y. Show that $f^{-1}(V)$ is closed.

(b) Answer any **two** : **4**

(i) Let $A = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$

Let $B = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$

Find d(A, B).

(ii) State (without proof) Urysohn's Lemma.

(iii) Show that any two closed and bounded intervals in $\mathbb{R}$ are homeomorphic.

(c) Answer **all** : **3**

(i) Give an example of a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$ which takes at least 2 different values. (i.e. the function is not constant). (Do not prove).

(ii) Give an example of a continuous function from S^1 to S' which takes at least 2 different values. (Do not prove). (Here $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$).

(iii) Give an example (without proof) of a function from $\mathbb{R}$ to $\mathbb{R}$ which is not continuous at 0.

4. (a) Let X be a metric space and let A be a subset of X.

Define : an open cover of A. Let K be a subset of X.

Define : K is compact. State (without proof) the Heine-Borel Theorem for $\mathbb{R}^n$. Show that any closed subset of a compact set, in a metric space X, is compact. **7**

OR

Let X be a metric space and let A be a subset of X.

Define : an open cover of A. Let K be a subset of X.

Define : K is compact. Show that the continuous image of a compact metric space is compact.

(b) Answer any **two** : **4**

(i) Show that a circle in $\mathbb{R}^2$ is not homeomorphic to a parabola.

(ii) Let $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$. Is A compact ?

(iii) Let $A = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$. Consider the function $f(x, y) = x^2 + y^2$. Find the maximum value of f on A.

(c) Answer **all** : **3**

(i) Give an example of a function $f : (0, 1) \to \mathbb{R}$, such that f is continuous but not bounded. (Do not prove).

(ii) Give a sequence (x_n) in $\mathbb{R}$ which has no convergent subsequence. (Do not prove).

(iii) Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

Let $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$

Find d(A, B). (Do not prove).

5. (a) Define : A metric space X is path connected. Show that $\mathbb{R}^2$ {(0, 0)} is path connected. **7**

OR

Define : A metric space X is connected.

Suppose X is connected and $f : X \to \{-1, 1\}$ is a continuous function.

Show that f is a constant function. Describe (without proof) the connected subsets of $\mathbb{R}$.

(b) Answer any **two** : **4**

(i) Give a path from (1, 0, 0) to (0, 1, 0) on S^2.

$(S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\})$

(ii) Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

(iii) Let $f(x) = x^2 - 3x + 3$, and let $I = [0, 2]$. Find f(I), the image of I under f.

(c) Answer **all** : **3**

(i) State the Intermediate Value Theorem.

(ii) Show that the polynomial $x^3 + x + 3$ has a real root.

(iii) Give an example of a polynomial with real coefficients, of degree at least one, which has no real root.