Seat No. : _______________

ND-140

December-2015

M.Sc., Sem.-III

502 : Mathematics

(Algebra-II)

Time : 3 Hours] **[Max. Marks : 70**

Instructions : (1) There are **five** questions.

(2) Each question carries equal **14** marks.

1. (a) Attempt any **one** : **7**

(1) If F is a field, prove that F[x] is a PID.

(2) If R is a commutative ring with unity, prove that R/A is integral domain $\Leftrightarrow$ A is a prime ideal.

(b) Attempt any **two** : **4**

(1) Determine all ring homomorphisms from Q to Q.

(2) Show that Q $(\sqrt{2})$ is not ring isomorphic to Q $(\sqrt{3})$.

(3) Let f[x] $\in$ $\mathbb{R}$ [x]. If f(a) = 0 for every integer a, prove that f(x) = 0 for all x.

(c) Attempt **all** : **3**

(1) Give an example of a subring that is not ideal.

(2) Give an example of a maximal ideal in $\mathbb{Z}_{30}$.

(3) If f(x) $\in$ $\mathbb{Z}$[x] is irreducible over Q, then it is irreducible over $\mathbb{Z}$. True or false. Justify.

2. (a) Attempt any **one** : **7**

(1) Let F be a field, p[x] ∈ f[x]. Prove that : p(x) is irreducible over F if and only if <p(x)> is maximal in F(x).

(2) State and prove modulo p irreducibility test.

(b) Attempt any **two** : **4**

(1) Prove that for every positive integer n, there are infinitely many polynomials in $\mathbb{Z}[x]$ of degree n that are irreducible over Q.

(2) If D is an integral domain and a, b ∈ D then show that a and b are associates if any only if <a> = <b>.

(3) Show that $\mathbb{Z}\left[\sqrt{-3}\right]$ is not PID.

(c) Attempt **all** : **3**

(1) Give an example of a field of order 4.

(2) Find the units of the ring $\mathbb{Z}\left[\sqrt{-3}\right]$.

(3) Prove that 3 is a prime but 13 is not a prime in $\mathbb{Z}$ [i].

3. (a) Attempt any **one** : **7**

(1) State and prove fundamental theorem of field theory (Kronecker's theorem).

(2) Let f(x) be irreducible over F and E be a splitting field of f(x) over F. Then prove that all the zeros of f(x) in E have the same multiplicity.

(b) Attempt any **two** : **4**

(1) Find the splitting field of $x^2 + x + 2$ over $\mathbb{Z}_3$.

(2) Prove that $Q\left(\sqrt{2}\,,\ \sqrt[3]{2}\right) = Q\left(\sqrt[6]{2}\right)$.

(3) Give an example to show that algebraic extension need not be finite extension.

(c) Attempt **all** : **3**

(1) Define algebraically closed field with an illustration.

(2) Describe the elements in Q (π).

(3) If a complex number Z is algebraic over Q, prove that $\sqrt{Z}$ is also algebraic over Q.

4. (a) Attempt any **one** : **7**

(1) For each divisor m of n, prove that GF(p^n) has a unique subfield of order p^m. Also these are the only subfields of GF(p^n).

(2) Prove that a regular g-gon can not be constructed with ruler and compass.

(b) Attempt any **two** : **4**

(1) Construct the field GF (2^3).

(2) Show that the non-zero elements of a finite field form a cyclic group under multiplication.

(3) If A and b $\neq$ 0 are constructible, prove that a/b is constructible.

(c) Attempt **all** : **3**

(1) Prove that the angle θ is constructible if and only if sin θ is constructible.

(2) Can we construct a field of order 15 ? Justify.

(3) If C is constructible, show that $\sqrt{|C|}$ is also constructible.

5. (a) Attempt any **one** : **7**

(1) If K = Q $\left(\sqrt[4]{2}\ ,\ i\right)$ and F = Q(i), find the Galois group Gal(K/F)-with details.

(2) If K = Q $\left(\sqrt{3}\ ,\sqrt{5}\right)$ and F = Q, find the Galois group Gal(K/F) with details.

(b) Attempt any **two** : **4**

(1) Define solvable group. Prove that A_n is solvable.

(2) Define n^{th} cyclotomic polynomial. Give first three cyclotomic polynomials.

(3) If $\alpha = \cos 2\pi/7 + i \sin 2\pi/7$, then what can be said about $Gal(Q(\alpha)/Q)$? Is it cyclic ? Explain.

(c) Attempt **all** : **3**

(1) If $n > 1$, prove that the product of n^{th} roots of unity is $(-1)^{n+1}$.

(2) State Galois theorem. (about the solvability by radicals implies solvable group)

(3) Is $ax^2 + bx + c$ solvable over Q ? Explain.