Seat No. : _______________

LD-110

April-2014

B.Sc. Sem. VI

CC-307 : Mathematics

(Abstract Algebra – II)

Time : 3 Hours] **[Max. Marks : 70**

Instructions : (1) **All** the questions are compulsory and carry **14** marks.
(2) Notations are usual, everywhere.
(3) Figures to the right indicate marks of the question/subquestion.

1. (a) Define a ring and a ring with unity. If R is a ring with unity, then prove that $(-1)\cdot a = -a$, for every $a \in R$ and prove also that $(-1)(-1) = 1$ in R. **7**

OR

Define a commutative ring and show that a ring R in which $(a + b)^2 = a^2 + 2ab + b^2$ holds true for all, $a, b \in R$ is a commutative ring.

(b) Define a Boolean ring and show that $(P(U), \Delta, \cap)$ is a Boolean ring. **7**

OR

Define an integral domain and prove that every field is an integral domain. Is the converse true ? Justify your answer in short.

2. (a) Define a subring and prove that a nonempty subset U of a ring R is a subring of R if and only if (i) $a - b \in U$ and (ii) $a\cdot b \in U$, for all $a, b \in U$. **7**

OR

Show that $I = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} / a, b \in Z \right\}$ is a left ideal which is not a right ideal of the ring $M_2(Z)$ of all 2×2 matrices with integral entries.

(b) Define an ideal. If R is a commutative ring with unity, which has no proper ideal, then prove that R is a field. **7**

OR

Define a ring Homomorphism. If $\Phi : (R, +, \cdot) \to (R', \oplus, \bullet)$ is a ring homomorphism then prove that $\Phi(I)$ is an ideal of R' whenever I is an ideal of R.

3. (a) Define the degree of a nonzero polynomial in $D[x]$.
For nonzero polynomials $f, g \in D[x]$, prove in usual notations that
$[f\cdot g] = [f] + [g]$ **7**

OR

If D' is the set of all constant polynomials in $D[x]$, then prove in usual notations that $D \cong D'$.

(b) Find the g.c.d. of $f(x) = 3x^3 + 2x^2 + 4$ and $g(x) = x^4 + 3x^2 + 1 \in Z_5[x]$ and express it into the form $a(x)\, f(x) + b(x)\, g(x)$. **7**

OR

State the Eisenstein's criterion and use it to prove that

$F(x) = x^n - p,\ n \geq 2$ is irreducible over Q.

4. (a) Define an extension field.

Also show that the set $Q[i] = \{a + i\, b \,/\, a, b \in Q\}$ is an extension field of Q. **7**

OR

Prove that an integral domain can be embedded into a field.

(b) If $I = < 4 >$, then show that I is a maximal but not a prime ideal of the ring 2Z of all even integers. **7**

OR

If I is a maximal idea of a commutative ring R with unity and if $a \in R$ with $a \notin I$, then show that $M = \{r{\cdot}a + k \,/\, r \in R \text{ and } k \in I\}$ is an ideal of R and M = R.

5. Attempt any **seven** of the following in **short** : **14**

(a) Give an example of a non-commutative infinite ring.

(b) Give an example of a finite commutative ring.

(c) Give an example of a non-Boolean ring.

(d) Give an example of a right ideal which is not a left ideal.

(e) Define maximal and prime ideal.

(f) Define an embedding of rings.

(g) Define any two of the following terms :

(i) Polynomial in D

(ii) $D[x]$

(iii) Quotient ring R/I.

(h) List the zeros of $f(x) = x^2 - 1$ in Z_{15}.

(i) State the factor theorem for polynomials.