Seat No. : _______________

TA-117

M.Sc. Sem.-II
April-2013

409 : Mathematics
(Complex Analysis – II)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Suppose f is analytic on an open disk $|z - z_0| < R_o$. Show that f(z) has the series representation **7**

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (|z - z_0| < R_0)$$

How are the coefficients given ?

OR

Suppose z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. If $R_1 = |z_1 - z_0|$, show that this power series converges uniformly in the closed disk $|z - z_0| \leq R_1$. Also show that the above power series represents a continuous function S(z) at each point inside the circle of convergence $|z - z_0| = R$.

(b) Attempt any **two** briefly : **4**

(1) Find the Laurent series for $f(z) = \dfrac{z+1}{z-1}$ which is valid in $1 < |z| < \infty$.

(2) Find the Laurent series in powers of z that represents the function $f(z) = \dfrac{1}{z(1+z^2)}$ in the domain $0 < |z| < 1$.

(3) Find the Laurent series in power of z represents the function $f(z) = \dfrac{1}{1+z}$ in the domain $1 < |z| < \infty$

(c) Answer very briefly : **3**

(1) Write the Laurent series for $f(z) = \exp\left(\dfrac{1}{z}\right)$ which is valid in $0 < |z| < \infty$.

(2) Express z^3 in powers of $(z - 1)$.

(3) Express e^z in powers of $(z - 1)$.

2. (a) Show that $\frac{p(z)}{q(z)}$ has a pole of order m if p and q are analytic at z_0, $p(z_0) \neq 0$ and q has a zero of order m at z_0. Derive that if m = 1, then z_0 is a simple pole of $\frac{p(z)}{q(z)}$ and $\underset{z = z_0}{\text{Res}} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$. **7**

OR

Show that an isolated singular point z_0 of a function f is a pole of order m if and only if f(z) can be written in the form $f(z) = \frac{\phi(z)}{(z - z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0. Also show in this case that $\underset{z = z_0}{\text{Res}} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$.

(b) Attempt any **two** briefly : **4**

(1) Find the value of the integral $\int_{|z| = 3} \frac{z^3 e^{\frac{1}{z}}}{1 + z^3} dz$.

(2) Find the value of the integral $\int_{|z| = 2} \frac{z^5}{1 - z^3} dz$.

(3) Find the value of the integral $\int_{|z| = 2} \tan z dz$.

(c) Answer very briefly : **3**

(1) Calculating a single residue of an appropriate related function find the value of $\int_{|z| = 2} \frac{1}{1 + z^2} dz$.

(2) What are all the singularities of the function $f(z) = \frac{1}{1 - z^4}$? Are they all isolated ?

(3) Find the value of the integral $\int_{|z| = 1} \exp\left(\frac{1}{z^2}\right) dz$.

3. (a) Write down the extension of Cauchy's Integral Formula and derive Cauchy's Inequality. Also state and prove Liouville's theorem. **7**

OR

Suppose f(z) is analytic and $| f(z) | \leq |f(z_0)|$ for each z in a neighbourhood $|z - z_0| < \epsilon$. Show that f(z) has the constant value $f(z_0)$ throughout the neighbourhood.

(b) Attempt any **two** briefly : **4**

(1) State carefully the Maximum Modulus Principle.

(2) Suppose that f is an entire function such that $| f(z) | \leq A | z |$ for all z, where A is a fixed positive constant. Show that $f(z) = a_1 z$ where a_1 is a complex constant.

(3) Suppose that f(z) is entire and that the harmonic function $u(x, y) = \text{Re}[f(z)]$ has an upper bound; that is $u(x, y) \leq u_0$ for all points (x, y) in the xy plane. Show that $u(x, y)$ must be constant throughout the plane.

(c) Answer very briefly : **3**

(1) Is f(z) = sin z a bounded function on C ? Justify using Liouville's theorem.

(2) Is the Minimum Modulus Principle (just like Maximum Modulus Principle) a valid statement ? Discuss.

(3) Suppose $f(z) = z^2$ and R is the closed triangular region determined by 1, i and 1 + i. Give geometric argument and figure also show the points on R, where the maximum and minimum of $| f(z) |$ occurs.

4. (a) Define the improper integral $\int_{-\infty}^{\infty} f(x)dx$ in two ways. Discuss, by giving appropriate proof or counter-example, the relationship between the two definitions. State the extra condition under which the two definitions are equivalent. Find the improper integral $\int_{0}^{\infty} \frac{x^2}{x^6 + 1} dx$. **7**

OR

Find the values of (i) $\int_{0}^{\infty} \frac{\cos ax}{x^2 + 1} dx \; (a > 0)$ (ii) $\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4 + 4} dx \; (a > 0)$

(b) Attempt any **two** briefly : **4**

(1) Evaluate $\int_{0}^{\infty} \frac{1}{(x^2 + 1)^2} dx$

(2) Evaluate $\int_{-\pi}^{\pi} \frac{1}{1 + \sin^2\theta} d\theta$

(3) Suppose that $| f(z) | \leq M_R$ on the semicircle $z = Re^{i\theta}$ $(0 \leq \theta \leq \pi)$ with $M_R \to 0$ as $R \to \infty$. Using Jordan's inequality show that $\lim_{R \to \infty} \int_{C_R} f(z)\, e^{iaz} dz = 0$

(c) Answer very briefly : **3**

(1) If $z = e^{i\theta}$, what is $\frac{z^3 + z^{-3}}{2}$?

(2) Evaluate $\int_0^\infty \frac{1}{x^2 + 1}\, dx$

(3) What is the value of $\operatorname*{Res}_{z=0} \frac{1}{2}$? Justify.

5. (a) Show that every linear fractional transformation, with one exception, has at most two fixed points in the extended complex plane. State clearly as to what is this exception ? Derive from this result that there is only one linear fractional transformation that maps three distinct points z_1, z_2, z_3 of the extended complex plane onto three distinct points w_1, w_2, w_3 of the extended complex plane respectively. **7**

OR

Find the linear fractional transformation that maps the points – i, 0, i onto the points –1, i, 1 respectively. Into what curve is the imaginary axis $x = 0$ transformed ?

(b) Attempt any **two** briefly. **4**

(1) Determine the number of roots (counting multiplicities) of the polynomial equation $2z^5 - 6z^2 + z + 1 = 0$ in the annulus $1 \leq |z| < 2$.

(2) Give detailed definition of the linear fractional transformation T from the extended complex plane $\mathbb{C} \cup \{\infty\}$ onto the extended complex plane $\mathbb{C} \cup \{\infty\}$. Describe its inverse. Is it also a linear fractional transformation ? Why ?

(3) Find the linear fractional transformation T which maps – i, 1, i onto – 1, 0, 1 respectively.

(c) Answer very briefly : **3**

(1) Determine the number of zeros (counting multiplicities) of the polynomial $z^4 - z^3 + z^2 - z + 5$ inside the circle $|z| = 1$.

(2) Find the winding number of the image of the unit circle under the map $f(z) = \frac{z^3 + 2}{z}$.

(3) Find the linear fractional transformation that maps 0, ∞, 1 onto ∞, 0, 1 respectively.