Seat No. : _______________

TF-116

M. Sc. (Sem. IV) Examination
May-2013

MATHEMATICS

511 EA (Differential Geometry – II)

Time : 3 Hours] **[Max. Marks : 70**

1. (A) Compute the second fundamental form of the hyperbolic paraboloid $\bar{\sigma}$ (u, v) = (u, v, $u^2 - v^2$). **7**

OR

Calculate the principal curvatures of the helicoid $\bar{\sigma}$(u, v) = (u cos(v), u sin(v), 2v), $0 < v < 2\pi$.

(B) Answer briefly any **two** : **4**

(i) Define a line of curvature on a surface S.
Give an example (without proof) of a line of curvature on the cylinder $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$.

(ii) Consider the xy-plane given by $\bar{\sigma}$ (u, v) = (u, v, o).
Let $\bar{r}$(t) be the circle on the xy-plane given by $\bar{r}(t) = \left(2 \cos\left(\frac{t}{2}\right), 2 \sin\left(\frac{t}{2}\right), 0\right)$.
Find its geodesic curvature on the xy-plane.

(iii) Define an umbilical point on a surface S. How many umbilical points are there on the ellipsoid $\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{4^2} = 1$? (Do not prove).

(C) Answer very briefly all **three** : **3**

(i) State (without proof) Meusnier's theorem.

(ii) State (without proof) Euler's theorem.

(iii) Define a hyperbolic point on a surface S.

2. (A) Calculate the Gaussian and mean curvatures of the surface $\bar{\sigma}$ (u, v) = (u, v, $u^2 + v^2$) at the point (1, –1, 2). **7**

OR

Calculate the Gaussian and mean curvatures of the surface

$\overline{\sigma}$ (u, v) = (u + v, u – v, uv) at the point (1, 1, 0).

(B) Answer briefly any **two** : **4**

(i) Show that the Gaussian curvature K of a ruled surface is negative or zero.

(ii) Define the Gauss map on an orientable surface S.

(iii) Find the image of the gauss map defined on the sphere

$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4\}$.

(C) Answer very briefly all **three** : **3**

(i) Express the principal curvatures in terms of the Gaussian curvature K and the mean curvature H. (Do not prove).

(ii) What is the Gaussian curvature of a sphere of radius R ? (Do not prove).

(iii) What is the curvature of a circle of radius R ? (Do not prove).

3. (A) State (without proof) Clairaut's theorem for geodesics on a surface of revolution S.

Show that every meridian on a surface of revolution is a geodesic.

Describe (without proof) the parallels on a surface of revolution which are geodesies. **7**

OR

Write down (without proof) necessary and sufficient conditions (geodesic equations) for the curve $\overline{r}(t) = \overline{\sigma}$ (u(t), v (t)) to be a geodesic. Show that an isometry between two surfaces takes the geodesics of one surface to the geodesics of the other.

(B) Answer briefly any **two** : **4**

(i) Describe (without proof) all geodesics on the circular cylinder $x^2 + y^2 = 1$.

(ii) Describe (without proof) all geodesics on the unit sphere.

(iii) Describe four different geodesics passing through the point (1, 0, 0) on the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$.

(C) Answer very briefly all **three** : **3**

(i) Define a geodesic on a surface S.

(ii) Describe (without proof) the geodesics on a plane.

(iii) What is the shortest path between the points (1, 0, 0) and (0, 1, 0) on the unit sphere ?

4. (A) State Gauss' Theorema Egregium. (Do not prove). Show that any map (of any part) of the earth's surface must distort distances. **7**

OR

State Gauss' Theorema Egregium. (Do not prove). State (without proof) a theorem completely describing compact surfaces with constant Gaussian curvature.

Does there exist any sphere which is isometric to the ellipsoid given by

$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{4^2} = 1$?

Prove your answer.

(B) Answer briefly any **two** : **4**

(i) Calculate the Christoffel symbols for the plane given by $\bar{\sigma}$ (u, v) = (u, v, o).

(ii) Does there exist an isometry between a sphere of radius 1 and a sphere of radius 2 ?

(iii) Give an example (without proof) of an isometry between two surfaces which is not obtained from a rigid motion of $\mathbb{R}^3$.

(C) Answer very briefly all **three** : **3**

(i) Suppose $\bar{\sigma} : U \to \mathbb{R}^3$ and $\bar{\subset} : U \to \mathbb{R}^3$ are surface patches with the same first and second fundamental forms. How are these surface patches related ?

(ii) Suppose $\bar{v}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ and $\bar{v}_2 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

Find $\bar{v}_3$, so that $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is a right-handed basis for $\mathbb{R}^3$.

(iii) Suppose $\{\bar{a}, \bar{b}, \bar{c}\}$ is a basis for $\mathbb{R}$, and suppose $\bar{d} = \alpha\bar{a} + \beta\bar{b} + \gamma\bar{c}$, $\alpha, \beta, \gamma \in \mathbb{R}^3$.

Find an expression for α in terms of $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ and the vector operations.

5. (A) State (without proof) the Gauss-Bonnet Theorem for compact surfaces. **7**

Suppose S is a compact surface with its Gaussian curvature K > O at every point. Show that S is diffeomorphic to a sphere.

OR

Define a critical point P of a smooth function $F : S \to \mathbb{R}$, where S is a surface. Define a non-degenerate critical point. Let $F : S \to \mathbb{R}$ be a smooth function on the unit sphere S with only finitely many critical points, all non-degenerate. If the number of local maxima of F is 12, and the number of saddle points of F is 30, what is the number of local minima of F ?

(B) Answer briefly any **two** : **4**

(i) Is the surface given below compact ?
$S = \{(x, y, z) : x^2 - y^2 + z^4 = 1\}$. Prove your result.

(ii) Give a vector field on the plane having multiplicity – 1 at the origin. (Do not prove).

(iii) Draw a vector field on a sphere with 1 source and 1 sink.

(C) Answer very briefly all **three**. **3**

(i) Define the Euler number χ of a triangulation of a compact surface S.

(ii) What is the Euler number of the torus ? (Do not prove).

(iii) Write down (without proof) the Euler number of the compact surface T_g of genus g.