Seat No. : _______________

AE-111

April-2015

B.Sc., Sem.-VI (CBCS System)

MAT-309 : Mathematics
Analysis – III

Time : 3 Hours] **[Max. Marks : 70**

Instructions : (1) **All** questions are compulsory.
(2) Each question carry equal marks.

1. (A) Let X be a complete metric space and let Y be a subspace of X. Prove that Y is complete if and only if it is closed.

OR

Let X be a metric space. A subset F of X is closed if and only if its complement F' is open.

(B) Define Metric space. Let X be a metric space. Then prove that
(1) Any union of open sets in X is open
(2) Any finite intersection of open sets in X is open

OR

Show that the set of all real numbers R with the distance from x to y defined by $d(x, y) = \frac{|x - y|}{1 + |x - y|}$ is a metric space.

2. (A) The metric space (X, d) is compact if and only if every sequence of points in X has subsequence converging to a point in X.

OR

Let X and Y be metric spaces and f a mapping of X into Y. Then f is continuous if and only if $f^{-1}(G)$ is open in X whenever G is open in Y.

(B) If f is a continuous mapping of a compact metric space X into a metric space Y, then f(X) is compact.

OR

Let X and Y be metric spaces and f a mapping of X into Y then prove that f is continuous at x_0 if and only if $f(x_n) \to f(x_0)$ whenever $x_n \to x_0$.

3. (A) Let (f_n) be a sequence of functions in R [a, b] converging uniformly to f. Then prove that $f \in R[a, b]$ and $\lim_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

OR

Find pointwise limit of $f_n(x)$ defined as follow if it exist. Determine whether $f_n(x)$ converges uniformly on the given set. Where $f_n(x) = \frac{x}{1 + nx}$; $x \in [0, 1]$.

(B) Let f_n satisfy

(i) $f_n \in D[a, b]$

(ii) $(f_n(x_0))$ converges for some $x_0 \in [a, b]$

(iii) f_n' Converges uniformly on [a, b]

then prove that f_n converges uniformly on [a, b] to a function f.

OR

Show that the series $\sum_{k=1}^{\infty} (xe^{-x})^k$ is uniformly convergent in [0, 2].

4. (A) State and prove Weierstrass Approximation theorem.

OR

State and prove Abel's limit theorem.

(B) If a function is represented by the power series of radius of convergence $r > 0$, then the power series is the Taylor's series of the function in the neighbourhood $\{x \in R : |x| < r\}$ of the origin.

OR

For every $x \in R$, and $n > 0$, prove that $\sum_{k=0}^{\infty} (nx - k)^2 \binom{n}{k} x^k (1 - x)^{n-k} = nx(1 - x) \le n/4$

5. Answer in short : (Any **seven**)

(1) Define open set.

(2) Define close set.

(3) Define uniform convergence.

(4) Let X be a non-empty set and let d be a real function of ordered pairs of elements of X which satisfies the condition $d(x, y) = 0 \Leftrightarrow x = y$. Is d metric on X ? Justify your answer.

(5) Explain Taylor's series.

(6) If $f_n(x) = \frac{1}{1 + nx}$ $(x \ge 0)$ then check continuity of $f_n(x)$.

(7) Is $f_n(z) = 1 + z + z^2 + \dots + z^{n-1}$ uniform convergent ?

(8) Show that the sum function $s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}$ is uniform continuous on [–1, 1] for $\alpha > 1$.

(9) Show that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ for $-1 \le x \le 1$.