Seat No. : _______________

AE-135

April-2015

M.Sc., Sem.-IV

Mathematics

MAT-507 : Functional Analysis-2

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any **one** : **7**

(i) If A is a positive operator on H, then prove that I + A is non-singular. Hence prove that I + T*T and I + TT* are non-singular for any T in B (H).

(ii) If T is an operator on H for which (Tx, x) = 0 for all x, then prove that T = 0.

(b) Attempt any **two** : **4**

(i) Let H = $\mathbb{R}^2$, K = $\mathbb{R}$ and A ∈ BL (H) is given by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that A is self-adjoint if and only if b = c.

(ii) Let H be a Hilbert space. A and B are self-adjoint, then prove that AB = 0 if and only if R(A) is orthogonal to R(B).

(iii) By an example, prove that $0 \le A \le B$ does not imply $A^2 \le B^2$.

(c) Answer very briefly. **3**

(i) Prove that $\| TT^* \| = \| T \|^2$.

(ii) Prove or disprove the mapping $y \to f_y$ form H to H* is linear.

(iii) Define positive operator. Give an example of self-adjoint operator that is not positive.

2. (a) Attempt any **one** : **7**

(i) If P is a projection on H with range M and null space N, then prove that $M \perp N \Leftrightarrow P$ is self-adjoint ; and in this case, $N = M^{\perp}$.

(ii) If P is the projection on M, then prove that

$$x \in M \Leftrightarrow Px = x \Leftrightarrow \| Px \| = \| x \|.$$

(b) Attempt any **two** : **4**

(i) Prove that the set of all normal operators is a closed subset of BL (H).

(ii) If P and Q are projections on closed linear subspaces M and N of H, then prove that Q – P is a projection if and only if $P \leq Q$.

(iii) Prove that right shift operator T on l^2 does not have eigen value.

(c) Answer very briefly : **3**

(i) If T = 4 I, then find the spectrum of T.

(ii) Define spectral resolution of T.

(iii) Give an example of a normal operator that is not self-adjoint.

3. (a) Attempt any **one** : **7**

(i) State and prove (briefly) finite dimensional spectral theorem.

(ii) Define the spectrum of the bounded operator on H. If H is finite dimensional complex Hilbert space then prove that spectrum of any bounded operator on H is non empty.

(b) Attempt any **two** : **4**

(i) If P and Q are projections on M and N respectively then under what condition P + Q is a projection ? Explain.

(ii) If x_1 and x_2 are eigen vectors corresponding to distinct eigen values λ_1 and λ_2 of T, prove that x_1 and x_2 are linearly independent.

(iii) Find the spectrum of operator P on $\mathbb{R}^2$ defined by P $(x, y) = (0, y)$.

(c) Answer very briefly : **3**

(i) Define similar matrices. Prove that similar matrices have the same determinant.

(ii) Can a 3×3 real matrix have empty spectrum ? Why ?

(iii) The operator T on R^2 defined by $T(x, y) = (x + y, x - y)$ is invertible. True / False ?

4. (a) Attempt any **one** : **7**

(i) State and prove Gelfand-Mazur theorem.

(ii) Let X be a Banach space then prove that the set G of all invertible operators in BL (X) is open and the inversion map is continuous on the set G.

(b) Attempt any **two** : **4**

(i) Find the spectrum of $A(x) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$ where $x \in l^p$.

(ii) Characterize the approximate eigen spectrum of T.

(iii) Find spectrum of right shift operator on l^2.

(c) Answer very briefly : **3**

(i) For A in BL (X) define eigen spectrum, approximate eigen spectrum and spectrum.

(ii) Find an operator A whose spectrum is [0, 1].

(iii) Find an operator A whose spectrum is {i}.

5. (a) Attempt any **one** : **7**

(i) Prove that $F \in BL(X, Y)$ is compact if and only if for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y.

(ii) Let X be a normed linear space and $A \in CL(X)$. If X is infinite dimensional then prove that 0 is an approximate eigen value of A.

(b) Attempt any **two** : **4**

(i) Prove that C L (X, Y) is a linear subspace of BL (X, Y).

(ii) Prove that every functional is a compact linear map.

(iii) By an example show that 0 can be a spectral value of a compact operator A without being its eigen value.

(c) Answer very briefly : **3**

(i) Give an example of a bounded linear operator that is not compact.

(ii) Prove or disprove a linear map T from $R^n \to R^m$ is compact.

(iii) True / False : The right shift operator on l^2 is compact.
