Seat No. : _______________

AI-104

April-2015

M.Sc., Sem.-IV

Mathematics

MAT-509-EA : Mathematical Methods

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any **one** : **7**

 (i) Solve : $xy'' + y' - xy = 0$

 (ii) Find the eigen values and eigen functions of the problem

 $y'' + \lambda y = 0,\ y(0) = 0,\ y'(L) = 0$

 (b) Attempt any **two** : **4**

 (i) Solve $y'' = y$ in terms of a power series in powers of $x - 1$.

 (ii) Using the indicated substitutions, reduce the following equation to Bessel's differential equation.

 $9x^2y'' + 9xy' + (36x^4 - 16)y = 0,\ (x^2 = z)$

 (iii) Show that $J_{-\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}}\cos x$.

 (c) Answer very briefly : **3**

 (i) Find the radius of convergence of $\sum\limits_{m=0}^{\infty} \dfrac{x^{2m+1}}{(2m+1)!}$.

 (ii) Show that $J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$.

 (iii) Show that $\int x^{-v} J_{v+1}(x)\, dx = -x^{-v} J_v(x) + c$.

2. (a) Attempt any **one** : **7**

 (i) Using the Laplace transform, solve the following initial value problem.

 $y'' - 5y' + 6y = 4e^t[1 - u(t - 2)],\ y(0) = 1,\ y'(0) = -2$

 (ii) State second shifting theorem. Using this theorem find the inverse Laplace transform of $\dfrac{3(1 - e^{-\pi s})}{s^2 + 9}$.

(b) Attempt any **two** : **4**

(i) Find the Laplace transform of $(t + 1)^2 e^t$.

(ii) Find the inverse Laplace transform of $\frac{1}{s(s^2 + w^2)}$.

(iii) Solve the integral equation :

$$y(t) = t + \int_0^t y(\tau) \sin(t - \tau) d\tau$$

(c) Answer very briefly : **3**

(i) Define unit step function and find its Laplace transform.

(ii) State the formula for the Laplace transform of the n^{th} derivative of a function f(t).

(iii) Find the Laplace transform of $te^{-t} \cos t$.

3. (a) Attempt any **one** : **7**

(i) Find the two half-range expansions of the function

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L - x) & \text{if } \frac{L}{2} < x < L \end{cases}$$

(ii) Show that

$$\int_0^\infty \frac{\cos xw + w \sin xw}{1 + w^2} dw = \begin{cases} 0 & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

(b) Attempt any **two** : **4**

(i) Find the Fourier cosine transform of

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } 1 < x < 2 \\ 0 & \text{if } x > 2 \end{cases}$$

(ii) Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

(iii) Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Further–more, let $f'(x)$ be absolutely integrable on the x-axis. Then prove that $\mathscr{F}\{f'(x)\} = iw\mathscr{F}\{f(x)\}$.

(c) Answer very briefly : **3**

(i) Is the following function $f(x)$, which is assumed to be periodic, of period 2π, even, odd or neither even nor odd ? Why ?

$$f(x) = \begin{cases} \cos^2 x & \text{if } -\pi < x < 0 \\ \sin^2 x & \text{if } 0 < x < \pi \end{cases}$$

(ii) What is the smallest positive period p of $\cos \frac{2\pi n x}{k}$?

(iii) Show that $\int_{-\pi}^{\pi} \cos^4 x \, dx = \frac{3\pi}{4}$

4. (a) Attempt any **one** : **7**

(i) Prove that if $Z[\{f(k)\}] = F(z)$, then $Z[\{a^k f(k)\}] = F\left(\frac{z}{a}\right)$. Using this formula, find the Z-transform of $c^k \sin \alpha k$, $k \geq 0$.

(ii) Solve the following differential equation by Z-transform

$6y_{k+2} - y_{k+1} - y_k = 0$, $y_0 = 0$, $y_1 = 1$

(b) Attempt any **two** : **4**

(i) Find the Z-transform of $\sin(3k + 5)$, $k \geq 0$.

(ii) Find the inverse Z-transform of $\frac{z}{(z-1)(z-2)}$ by residue method.

(iii) State and prove initial value theorem.

(c) Answer very briefly : **3**

(i) What is the inverse Z-transform of $\frac{4z}{z-a}$, when $|z| > |a|$?

(ii) What is the order of the differential equation $y_{k+1} - 2y_{k-1} = k^2$?

(iii) State final value theorem.

5. (a) Attempt any **one** : **7**

(i) Show that if $n = 0$, the Hankel transform :

$$H\left\{\frac{\sin ax}{x}\right\} = \begin{cases} 0 & \text{if } s > a \\ \frac{1}{\sqrt{a^2 - s^2}} & \text{if } 0 < s < a \end{cases}$$

(ii) Show that $\int_0^a x(a^2 - x^2) J_0(sx) dx = \frac{4a}{s^3} J_1(as) - \frac{2a^2}{s^2} J_0(as)$.

(b) Attempt any **two** : **4**

(i) Find the Hankel transform of zero order of $\frac{e^{-ax}}{x}$.

(ii) Find $H^{-1}[e^{-as}]$, when n = 0

(iii) Find the Hankel transform of the function

$$f(x) = \begin{cases} x^n & \text{if } 0 < x < a, \ n > -1 \\ 0 & \text{if } x > a, \quad n > -1 \end{cases}$$

(c) Answer very briefly : **3**

(i) What is the value of the integral $\int_0^\infty e^{-ax} J_0(sx)\, dx$?

(ii) Show that Hankel transform is a linear operation.

(iii) Find $H[e^{-ax}]$, n = 1