Seat No. : _______________

AK-120

April-2015

M.Sc., Sem.-IV

Mathematics

MAT-511-EA – Differential Geometry – II

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Calculate the principal curvatures of the catenoid **7**

$\sigma(u, v) = (\cosh(u)\cos(v), \cosh(u)\sin(v), u)$,

$-\infty < u < \infty,\ 0 < v < 2\pi$

OR

Calculate the principal curvatures of the elliptic paraboloid

$\sigma(u, v) = (u, v, u^2 + v^2)$,

$-\infty < u < \infty,\ -\infty < v < \infty$

(b) Answer any two : **4**

(i) State Meusnier's theorem. (Do not prove)

(ii) State Euler's theorem. (Do not prove)

(iii) Define a line of curvature on a surface S.

(c) Answer **all** : **3**

(i) Suppose that every point of a connected surface S is an umbilic. Without proof, describe the surface S.

(ii) Give an example of a parabolic point on a surface. Do not prove.

(iii) Calculate the second fundamental form of the plane

$\sigma(u, v) = a + up + vq$

Where a, p, q are fixed and p and q are orthogonal unit vectors.

2. (a) Calculate the Gaussian and mean curvatures of the surface **7**

$\sigma(u, v) = (u, v, uv)$ at the point (2, 3, 6),

$-\infty < u < \infty,\ -\infty < v < \infty$

OR

Calculate the Gaussian and mean curvatures of the surface

$z = \sin(x) + \cos(y)$

(b) Answer any **two** : **4**

(i) What is the image under the Gauss map of the plane $x + y + z = 0$?

(ii) What is the image under the Gauss map of the unit sphere ?

(iii) Show that the surface given by

$x^2 + y^2 + z^4 = 1$

has at least one elliptic point. (State the relevant results)

(c) Answer **all** : **3**

(i) Give an example of a surface at every point of which the Gaussian curvature is 4.

(ii) Name a surface at every point of which the Gaussian curvature is negative.

(iii) Name a compact surface having some points at which the Gaussian curvature is negative.

3. (a) Find the geodesics on a circular cylinder. Find the length of the shortest path between the points (0, 1, 0) and $\left(-1, 0, \frac{\pi}{2}\right)$ on the cylinder $x^2 + y^2 = 1$. **7**

OR

Show that an isometry between two surfaces takes the geodesics of one surface to the geodesics of the other.

Describe (without proof) the geodesics of the unit sphere.

Describe (without proof) the isometries of the unit sphere to itself.

(b) Answer any **two** : **4**

(i) State Clairaut's theorem. (Do not prove)

(ii) Show that any geodesic has constant speed.

(iii) Find the length of the shortest path on the unit sphere between the points $\left(\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}\right)$ and (1, 0, 0).

(c) Answer **all** : **3**

(i) Show that any straight line on a surface is a geodesic.

(ii) Find the length of the shortest path on the plane $x + y + z = 0$, between the points (1, 1, –2) and (–1, 1, 0).

(iii) Give a geodesic on the surface $z = x^2 - y^2$ which passes through the point (0, 0, 0).

4. (a) State Gauss's Theorema Egregium. Show that there is no isometry between any region of a sphere and any region of a plane. Show that there is no isometry between any region of the unit sphere and any region of the surface given by

$\sigma(u, v) = (u, u^2, v), -\infty < u < \infty,$

$-\infty < v < \infty$ **7**

OR

State Gauss's Theorema Egregium. Show that there is no surface patch whose first and second fundamental forms are $du^2 + dv^2$ and $\cos^2(u)\, du^2 + dv^2$ respectively.

(b) Answer any **two** : **4**

(i) A compact surface has constant Gaussian curvature equal to 2. Describe the surface.

(ii) If E = 1 and F = 0, give a formula for the Gaussian curvature in terms of $\sqrt{G}$ and its derivatives. (Do not prove)

(iii) Give an isometry of the helicoid $\sigma(u, v) = (u \cos(v)\ u \sin(v), v)$, other than the identity. (Do not prove)

(c) Answer **all** : **3**

(i) Give a rigid motion of the plane Z = 0, other than the identity.

(ii) Calculate the Christoffel symbols for the surface given by $\sigma(u, v) = (u, 0, v)$.

(iii) A surface of revolution is given by $\sigma(u, v) = (f(u) \cos (v), f(u) \sin(v), g(u))$, where $f > 0$ and $\left(\frac{df}{du}\right)^2 + \left(\frac{dg}{dv}\right)^2 = 1$.

Write down (without proof) a formula for the Gaussian curvature in terms of f and its derivatives.

5. (a) Show that, if a compact surface S is diffeomorphic to the torus T_1, then **7**

$$\iint_S K\, dA = 0.$$

Draw a vector field on the torus T_1 with no stationary points.

OR

Suppose that S is a compact surface whose Gaussian curvature K is strictly greater than zero everywhere. Show that S is diffeomorphic to a sphere.

Draw a picture of a surface which is diffeomorphic to a sphere but has some points of negative Gaussian curvature.

(b) Answer any **two** : **4**

(i) Give an example of a smooth tangent vector field on the plane which has multiplicity 1 at the origin.

(ii) Give an example of a smooth tangent vector field on the plane which has multiplicity –1 at the origin.

(iii) Show that a smooth tangent vector field on a sphere has at least one stationary point.

(c) Answer **all** : **3**

(i) Is the surface $x^2 + y^2 - z^2 = 1$ compact ?

(ii) Let S be the unit sphere. Consider $f : S \to \mathbb{R}$ given by $f(x, y, z) = z$. How many local maxima and local minima does f have ? (Do not prove)

(iii) Let $f(x) = \sin(3x)$, $0 < x < 2\pi$. How many local maxima does f have ? (Do not prove)
