Seat No. : _______________

11C-129

May-2015

M.Sc. Sem.-II

408 : Mathematics

(Algebra-I)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any **one** : **7**
   (i) State and prove Cayley's theorem.
   (ii) State and prove Orbit-Stabilizer theorem.

   (b) Attempt any **two**. **4**
   (i) Show that a function from a finite set S to itself is one-one if and only if it is onto. Is this true when S infinite ?
   (ii) Prove that the the mapping from U(16) to itself given by $x \rightarrow x^3$ is an automorphism.
   (iii) Let G be a group and H be a subgroup of G. Let $a \in G$. Prove that aH = H if and only if $a \in H$.

   (c) Answer very briefly. **3**
   (i) How many elements of order 5 are in S_7 ?
   (ii) Prove or disprove : $Aut(Z_{25})$ is cyclic.
   (iii) If $|G| = 8$, then prove that there is an element $a \neq e$ such that $a^2 = e$.

2. (a) Attempt any **one**. **7**
   (i) State and prove G/Z theorem.
   (ii) Let G be a finite abelian group and let p be a prime that divides the order of G, then prove that G has an element of order p.

   (b) Attempt any **two** : **4**
   (i) Prove that $Inn(D_6)$ is isomorphic to D_3.
   (ii) Find the number of element of order 10 in $Z_{25} \oplus Z_{100}$.
   (iii) Give an example of a group G with a proper subgroup H such that G and H are isomorphic.

   (c) Answer very briefly. **3**
   (i) True/False : U(7) is isomorphic to U(9).
   (ii) Find the last two digits of 99^{99}.
   (iii) Prove that (R, +) is not isomorphic to (Q, +).

3. (a) Attempt any **one**. **7**
   (i) Define the internal direct product of $H_1, H_2, \ldots, H_n$. Prove that the internal direct product is isomorphic to external direct product.
   (ii) State (only) Fundamental theorem of finite Abelian group. Find the isomorphism class of U(24).

(b) Attempt any **two** : **4**

(i) Find all abelian groups (upto isomorphism) of order 72.

(ii) Prove or disprove : If G is a group, then $H = \{g^2/g \in G\}$ is a subgroup of G.

(iii) What is the smallest positive integer n such that there are exactly 4 non-isomorphic Abelian groups of order n ?

(c) Answer very briefly. **3**

(i) Let $G = D_8$, find a homomorphism from G to the group G' = {–1, 1} with multiplication.

(ii) If f is a homomorphism from group G to group G', prove that the order of f(g) divides the order of g.

(iii) State (only) First Isomorphism Theorem.

4\. (a) Attempt any **one**. **7**

(i) If H is subgroup of finite group of G and |H| is a power of p, then prove that H is contained in some Sylow p-subgroup of G.

(ii) Let G be a finite group and let $a \in G$. Then show that Cl(a) = |G : C(a)|. Use it to show that $|G| = \sum |G : C(a)|$ where the sum runs over one element from each conjugacy class of G.

(b) Attempt any **two** : **4**

(i) Prove that a unique Sylow p-subgroup is normal.

(ii) Show that Cl(a) = {a} if and only if $a \in Z(G)$.

(iii) How many non-isomorphic groups of order 35 are possible ? Justify.

(c) Answer very briefly. **3**

(i) If order of G is 21, find all Sylow 7-subgroups of G.

(ii) True/False : The group of order 169 is abelian.

(iii) Define the term: Conjugate subgroup.

5\. (a) Attempt any **one**. **7**

(i) State Index theorem and Embedding theorem. Prove anyone of the two theorems.

(ii) Define Simple group. Prove that A_5 is simple.

(b) Attempt any **two.** 4

(i) Does there exist a simple group of order 100 ?

(ii) Prove or disprove : If the order of group G is *pqr* where *p, q, r* are distinct primes then G can not be simple.

(iii) Show that A_5 cannot have a subgroup of order 30.

(c) Answer very briefly. **3**

(i) State (only) Burnside theorem.

(ii) If the order of group G is 43, then prove that G is simple.

(iii) Determine the number of ways in which the four corners of a square can be coloured with two colours. (you can use a single colour on all four corners.)

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