Seat No. : ______________

18I-101

May-2015

M.Sc., Sem.-II

411 : Mathematics

(Real Analysis)

Time : 3 Hours] **[Max. Marks : 70**

1. (A) Attempt any **one**. **7**

   (1) If the sequence of measurable functions $f_n(x)$ converges to $f(x)$ almost everywhere on a bounded measurable set E, then prove that $f_n \Rightarrow f$.

   (2) State and prove Riesz theorem.

   (B) Attempt any **two**. **4**

   (1) Verify Egorov's theorem for the sequence $f_n : [0, 1] \to R$ defined by $f_n(x) = 2x^n + 1$.

   (2) If $f_n \Rightarrow f$ and $g_n \Rightarrow g$, then show that $f_n + g_n \Rightarrow f + g$.

   (3) State (only) Luzin's theorem.

   (C) Answer in brief. **3**

   (1) True or False : If $f_n \Rightarrow f$, then every subsequence of $\{f_n\}$ converges in measure to f.

   (2) If $f_n \Rightarrow f$ and $f_n \Rightarrow g$, then show that f = g almost everywhere.

   (3) Define : Convergence in measure.

2. (A) Attempt any **one**. **7**

   (1) Define Bernstein polynomial. If $f(x)$ is a continuous function on [0, 1], then prove that the sequence of its Bernstein polynomials converges uniformly to f on [0, 1].

   (2) Prove that $L_p[a, b]$ is complete.

(B) Attempt any **two**. **4**

(1) If f, g $\in L_p[a, b]$, then show that $f + g \in L_p[a, b]$.

(2) Show that the set of bounded measurable functions is dense in $L_p[a, b]$.

(3) State (only) Holder's inequality for functions as well as numbers.

(C) Answer in brief. **3**

(1) Express $\cos^2 x$ in the form of a trigonometric polynomial.

(2) If $f_n \to f$ in $L_p[a, b]$, then show that $\| f_n \|_p \to \| f \|_p$.

(3) True or False : $L_p[a, b] \subset L_1[a, b]$ for all $p > 1$.

3. (A) Attempt any **one**. **7**

(1) Show that if $f : [a, b] \to R$ is of finite variation and continuous at x_0, then the function $\pi(x) = V_a^x(f)$ is continuous at x_0.

(2) If $f : [a, b] \to R$ is increasing, then show that its derivative $f'(x)$ is measurable and $\int_a^b f'(x)\, dx \le f(b) - f(a)$.

(B) Attempt any **two**. **4**

(1) Let $f(x) = x \sin(1/x)$ for $(x \neq 0)$ and $f(0) = 0$. Then compute any three derived numbers of f at 0.

(2) If $f(x) = \tan x$, then compute the total variation of f on $[0, \pi/4]$.

(3) If f is of finite variation on R, then show that $\lim_{x \to \infty} V_x^{\infty}(f) = 0$.

(C) Answer in brief : **3**

(1) Give the definition of saltus function.

(2) How do we define the total variation of a real-valued function defined on R ?

(3) True or False : Every function of finite variation on [a, b] is continuous.

4. (A) Attempt any **one**. **7**

(1) If f : [a, b] $\to$ R is such that $f'(x)$ is finite everywhere and summable on [a, b], then prove that

$$f(c) = f(a) + \int_a^c f'(t)dt, \quad a < c \le b.$$

(2) Let $f(x) = x^2 \cos(\pi/x^2)$ and $g(x) = x^{3/2} \sin(1/x)$ for ($x \ne 0$) and $f(0) = g(0) = 0$. Show that f is not Lebesgue integrable on [0, 1] but g is Lebesgue integrable on [0, 1].

(B) Attempt any **two**. **4**

(1) Let f be summable on [a, b] and $\phi(x) = \int_a^x f(t)\, dt$. If f is continuous at x_0, then show that $\phi'(x_0) = f(x_0)$.

(2) Show that every Lipschitz continuous function on [a, b] is absolutely continuous.

(3) Prove that $f(x) = x^2 + |x|$ is absolutely continuous on [–1, 1].

(C) Answer in brief. **3**

(1) True or False : Every continuously differentiable function on [a, b] is absolutely continuous function.

(2) State Vitali's covering lemma.

(3) Explain what we mean by a Lebesgue point of a summable function.

5. (A) Attempt any **one**. **7**

(1) For $f \in L_2[-\pi, \pi]$, if $S_N(x)$ denotes the partial sums of the Fourier series of f, then show that $\|f - T_N\|_2 \ge \|f - S_N\|_2$, for every trigonometric polynomial T_N of degree N.

(2) Determine the Fourier series of the 2π periodic function

$$f(t) = \begin{cases} 0 & -\pi \le t \le 0 \\ 1 & 0 < t < \pi. \end{cases}$$

Deduce from it the value of the infinite sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$.

(B) Attempt any **two**. **4**

(1) State and prove Riemann-Lebesgue lemma.

(2) Show that if the series $\sum c_n$ is cesaro-summable and $nc_n \to 0$, then $\sum c_n$ is summable (convergent).

(3) State and prove Bessel's inequality for $f \in L_2 [-\pi, \pi]$.

(C) Answer in brief : **3**

(1) Define $D_N(x)$ and determine its value when x is a multiple of 2π.

(2) Show that $\frac{1}{\pi} \int_{-\pi}^{\pi} F_N(x)\, dx = 1$.

(3) Can we say that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$ is a Fourier series for some function in $L_2[-\pi, \pi]$? Why ?
