Seat No. : _______________

13E-109

May-2015

M.Sc., Sem.-II

409 : Mathematics

(Complex Analysis – II)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Suppose z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. If $R_1 = |z_1 - z_0|$, show that this power series converges uniformly in the closed disk $|z - z_0| \leq R_1$. Also show that the above power series represents a continuous function S(z) at each point inside the circle of convergence $|z - z_0| = R$ **7**

OR

Suppose f is analytic on an open disk $|z - z_0| < R_0$. Show that f(z) has the series representation.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \left(|z - z_0| < R_0\right)$$

where $a_n = \frac{f^n(z_0)}{n!}$.

(b) Answer any **two** of the following briefly : **4**

(i) Specifying the domains, give two Series Expansions in powers of z for the function $f(z) = \frac{1}{(z-2)(1-z)}$.

(ii) Specifying the domains, give two Laurent series Expansions in powers of z for the function $f(z) = \frac{1}{z(1+z^2)}$.

(iii) When do you say that $\lim_{n \to \infty} zn = z$? Show that $\lim_{n \to \infty} z_n = z$ if and only if $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$.

(c) Answer **all** of the following very briefly : **3**

(i) Represent $f(z) = 1 + z^2$ in powers of $z - 5$.

(ii) Find the Maclaurin Series Expansion of $f(z) = \frac{z}{z^4 + 9}$ which is valid in $|z| < \sqrt{3}$.

(iii) Discuss the phrase : *"Circle of Convergence"*.

2. (a) Describe three types of isolated singular points with an illustration of each type. **7**

OR

Show that an isolated singular point z_0 of a function f is a pole of order m if and only if f(z) can be written in the form $f(z) = \frac{\phi(z)}{(z - z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0. Also show in this case that $\underset{z = z_0}{\text{Res}}\, f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$.

(b) Answer any **two** of the following briefly : **4**

(i) Show that $\underset{z = i}{\text{Res}} \frac{\text{Log } z}{(z^2 + 1)^2} = \frac{\pi + 2i}{8}$

(ii) Write the principal parts for the following functions at their isolated singular points, also assert the type of the singularity.

(I) $\frac{\cos z}{z}$

(II) $\frac{1}{(2 - z)^3}$

(iii) Suppose that $z_0 = \sqrt{2}e^{i\frac{\pi}{4}} = 1 + i$. Find $\underset{z = z_0}{\text{Res}} \frac{z}{z^4 + 4}$

(c) Answer all of the following very briefly : **3**

(i) Find $\underset{z = 0}{\text{Res}} \frac{1}{z^2}$.

(ii) Find the value of the integral $\int_{|z| = 1} \exp\left(\frac{1}{z^2}\right) dz$.

(iii) Describe all the singular points of $\frac{1}{\sin\left(\frac{\pi}{z}\right)}$. Which of these are isolated singular points and which are not ?

3. (a) State and prove Liouville's theorem. Derive, after carefully stating, the Fundamental Theorem of Algebra. **7**

OR

Suppose f(z) is analytic and $|f(z)| \leq |f(z_0)|$ on $|z - z_0| < \in$. Show that f is constant throughout the neighbourhood.

(b) Answer any **two** of the following briefly : **4**

(i) Suppose that f(z) is entire and that the harmonic function $u(x, y) = Re[f(z)]$ has an upper bound; that is $u(x,y) \leq u_0$ for all points (x, y) in the xy plane. Show that $u(x, y)$ must be constant throughout the plane.

(ii) Let $f(z) = (z - i)^2$ and R be the closed triangular region determined by 0, –1 and – 2i. Give geometric argument and determine the points on R where the maximum and minimum of $|f(z)|$ occurs.

(iii) Suppose $f(z) \neq 0$ is continuous on a closed region R. Show that $|f(z)|$ has a minimum value m in R which occurs on the boundary of R and never in the interior.

(c) Answer **all** of the following very briefly : **3**

(i) Is the function sin z, $z \in \mathbb{C}$ a bounded function? Justify.

(ii) What are the maximum and minimum values of $f(z) = |\exp z|$ on the rectangular region R described by $0 \leq x \leq 1$, $0 \leq y \leq \pi$? Where are they attained ?

(iii) Is it true that $|f(z)|$ can have its minimum value at an interior point of R ? Justify.

4. (a) State the appropriate assumption of Jordan's lemma. Under these assumptions, show that :

$$\lim_{R \to \infty} \int_{C_R} f(z)e^{iaz}\, dz = 0$$

OR

Evaluate the integral $\displaystyle\int_0^{\infty} \frac{x^2 dx}{(x^2 + 9)(x^2 + 4)^2}$ giving all the details. **7**

(b) Answer any **two** of the following briefly : **4**

(i) Giving the main steps only and using residue theory, show that

$$\int_0^{\infty} \frac{x \sin 2x}{x^2 + 3}\, dx = \frac{\pi}{2} \exp(-2\sqrt{3})$$

(ii) Giving the main steps only and using residue theory, show that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 2x + 2}\, dx = \frac{\pi}{e}(\sin 1 + \cos 1)$$

(iii) Giving the main steps only and using residue theory, show that

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

(c) Answer **all** of the following very briefly : **3**

(i) Discuss the Phrase: "Winding Number".

(ii) Calculate Δ_C arg f(z) where C is $|z| = 1$ and $f(z) = z^2$.

(iii) Define the improper integral $\int_{-\infty}^{\infty} f(x)\, dx$ in two different ways. Show that the two definitions are not equivalent.

5. (a) State the conditions under which f(z) and f(z) + g(z) have the same number of zeros counting multiplicities inside a simple closed contour C. Derive the Fundamental Theorem of Algebra using this result. **7**

OR

Define very carefully Möbius Transformation as a bijection from the extended complex plane onto the extended complex plane. Find explicitly the inverse of a Möbius Transformation and state clearly as to why it is also a Möbius Transformation. Also show that composition of two Möbius Transformations is also a Möbius Transformation.

(b) Answer any **two** of the following briefly : **4**

(i) Find the linear fractional transformation T which maps 1, 0, –1 onto i, ∞, 1 respectively.

(ii) Find the linear fractional transformation T which maps –i, 0, i onto – l, i, 1 respectively.

(iii) Determine the number of roots (counting multiplicities) of the polynomial equation $z^5 + 3z^3 + z^2 + 1 = 0$ inside the circle $|z| = 2$.

(c) Answer **all** of the following very briefly : **3**

(i) Find the winding number of the image of $|z| = 1$ under the map $f(z) = \frac{(2z-1)^7}{z^3}$.

(ii) Give an example of Möbius Transformation which has exactly one fixed point.

(iii) What is the winding number of the image of the unit circle under the map $w = \frac{1}{z^2}$? Justify.
