Seat No. : _______________

XB-125

T.Y.B.Sc.

March-2013

Mathematics : Paper – VIII

(Analysis – II)

Time : 3 Hours] **[Max. Marks : 105**

1. (a) Attempt any **three :** **18**

(1) Prove that monotonic decreasing bounded below sequence is convergent.

(2) Let (S_n) be a bounded sequence of real numbers. If $\lim_{n\to\infty} \sup S_n = M$ then prove that for any $\in > 0$.

(i) $S_n < M + E$ for all except a finite number of values of n.

(ii) $S_n > M - \in$ for infinitely many values of n.

(3) If $\lim_{n\to\infty} t_n = M$ where $M \neq 0$, then prove that $\lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{M}$

(4) If $S_1 = \sqrt{2}$ and $S_{n+1} = \sqrt{2}\sqrt{S_n}$ for $n \geq 1$, prove that (S_n) is a monotonic increasing sequence bounded above and $\lim_{n\to\infty} S_n = 2$.

(5) Prove that the sequence defined by the relation $S_{n+2} = \frac{1}{2}(S_{n+1} + S_n)$ converges provided that $S_1 \neq S_2$.

(b) Find the $\lim_{n\to\infty} \sup S_n$ and $\lim_{n\to\infty} \inf S_n$ for the sequence $S_n = (-1)^n \left(1 + \left(\frac{1}{n}\right)\right)$. **3**

2. (a) State and prove Weierstrass M-test. **6**

OR

If $\sum a_n$ is a series of non-negative numbers which converges to $A \in R$ and $\sum b_n$ is rearrangement of $\sum a_n$, then prove that $\sum b_n$ is convergent and $\sum b_n = A$.

(b) Attempt any **two :** **12**

(1) Prove that $1 + \frac{1}{2\,!} + \frac{1}{4\,!} + \frac{1}{6\,!} + \ldots$ converges.

(2) Prove that the series $\sum(-1)^n \left[\sqrt{n^2 + 1} - n\right]$ is conditionally convergent.

(3) Discuss the uniform convergence of sequence of function $f_n(x) = \frac{nx}{1 + n^2x^2}$ $(-\infty < x < \infty)$.

(c) Discuss the convergence of $\sum_{n=2}^{\infty} \frac{1}{n \log n}$. **3**

3. (a) Attempt any **three :** **18**

(1) If $g \in R\,[a, b]$ then prove that $\frac{1}{g} \in R\,[a, b]$, where g is bounded away from zero.

(2) State and prove first fundamental theorem of calculus.

(3) Let g be continuous function on [a, b] and f has a derivative which is continuous and never changes sing on [a, b]. Then prove that for some

$$C \in [a, b] \int_a^b f(x)\, g(x)\, dx = f(a) \int_a^c g(x)\, dx + f(b) \int_c^b g(x)\, dx.$$

(4) If $f \in R\,[a, b]$ then prove that $|\,f\,| \in R\,[a, b]$. Also, prove that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |\,f(x)\,|\, dx$$

(5) Prove that

$$\frac{\pi^3}{24} \leq \int_0^{\pi} \frac{x^2}{5 + 3\cos x}\, dx \leq \frac{\pi^3}{6}$$

(b) Give an example of a function which is bounded on [a, b] but not Riemann integrable. **3**

4. (a) Let f be a non-increasing function on $[1, \infty)$ such that $f(x) \geq 0$ for $1 \leq x < \infty$. Then prove that $\sum_{n=1}^{\infty} f(n)$ converges if $\int_1^{\infty} f(x)dx$ converges and $\sum_{n=1}^{\infty} f(n)$ diverges if $\int_1^{\infty} f(x)dx$ diverges. **6**

OR

(a) Test for convergence :

(1) $\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$

(2) $\int_0^{\infty} \frac{1}{x^3 + x^{1/3}}\, dx$

(b) Attempt any **two :** **12**

(1) Let $f(x) = \sum a_n x^n$ be a power series with radius of convergence 1. If the series converges at 1, then prove that $\lim_{x \to 1^-} f(x) = f(1)$.

(2) Prove that for $-1 < x \leq 1$

$$\frac{1}{2}(\tan^{-1}x)^2 = \frac{x^2}{2} - \frac{x^4}{4}\left(1 + \frac{1}{3}\right) + \frac{x^6}{6}\left(1 + \frac{1}{3} + \frac{1}{5}\right) + \ldots.$$

(3) State and prove Weierstrass Approximation theorem.

(c) Discuss the uniform convergence of $f_n(x) = \dfrac{1}{1 + nx}$, $0 \leq x \leq 1$. **3**

5. (a) Attempt any **three** : **18**

(1) State and prove sufficient conditions for existence of that derivative of a function w = f(z) at a point $z_0 = (x_0, y_0)$.

(2) Find the image of the infinite strip $0 < y < \dfrac{1}{(2c)}$, $c \neq 0$ under the transformation $w = \dfrac{1}{z}$. Sketch the strip and its image.

(3) Find the harmonic conjugate of sinh x sin y and corresponding analytic function in terms of z.

(4) Verify conformality of $w = z^2$ by considering the curves $y = 2x$ and $y = x - 1$ and their images.

(5) Find the image of the curve $|z| = 2$ under the mapping $w = z + \dfrac{1}{z}$, $z \neq 0$.

(b) Find the non-conformal points of the transformation $w = 2z^3 - 21z^2 + 72z + 9$. **3**
