Seat No. : _______________

JD-111

July-2021

B.Sc., Sem.-VI

309 : MATHEMATICS
(Analysis-III)

Time : 2 Hours] **[Max. Marks : 50**

Instruction : (1) Attempt any **three** questions in **Section-I**.
(2) **Section-II** is a **compulsory** section of short questions.
(3) Notations are usual everywhere.
(4) The right hand side figures indicate marks of the sub question.

SECTION – I

Attempt any **THREE** of the following questions :

1. (A) Define a metric space. Prove that a subset G of a metric space X is open if and only if it is a union of open spheres. **7**

   (B) Let X and Y be metric spaces and f a mapping of X into Y then prove that f is continuous if and only if f^{-1} (G) is open in X whenever G is open in Y. **7**

2. (A) Prove that a subset F of a metric space X is closed in X if and only if its complement F' is open in X. **7**

   (B) Show that a real function d defined on ordered pairs of elements of a nonempty set X satisfying the conditions

   $d(x, y) = 0 \Leftrightarrow x = y$, and $d(x, y) \leq d(x, z) + d(y, z)$ is a metric on X. **7**

3. (A) Prove that Compact subsets of metric spaces are closed. **7**

   (B) Let X and Y be metric spaces and f a mapping of X into Y then prove that f is continuous at x_0 if and only if $f(x_n) \to f(x_0)$ whenever $x_n \to x_0$. **7**

4. (A) Prove that closed subsets of a compact set are compact in a metric space. **7**

   (B) A mapping f of a metric space X into a metric space Y is continuous on X if and only if f^{-1} (V) is open in X for every open set V in Y. **7**

5. (A) State and prove Weierstrass M-test. Show that $f_n(x) = n^2 x^n (1 - x); x \in [0, 1]$ converges pointwise to a function which is continuous on [0, 1]. **7**

(B) Let (f_n) be a sequence of functions in R[a, b] converging uniformly to f.

Then $f \in R[a, b]$ and $\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$. **7**

6. (A) Let (f_n) be a sequence of continuous function of $E \subset C$ converges uniformly to f on E then prove that f is continuous on E. **7**

(B) Let f_n satisfy (1) $f_n \in D[a, b]$ (2) $(f_n(x_0))$ converges for $x_0 = \in D[a, b]$

(3) f_n converges uniformly on [a, b] then prove that f_n converges uniformly on [a, b] to a function f. **7**

7. (A) State and prove Abel's limit theorem. **7**

(B) Show that for $-1 \leq x \leq 1$, $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1} \frac{x^n}{n} + \cdots$.

Hence evaluate log2. **7**

8. (A) For every $x \in R$ and $n > 0$

Prove that $\Sigma_{k=0}^{n} (nx - k)^2 \binom{n}{k} x^k (1 - x)^{n-k} = nx(1 - x) \leq \frac{n}{4}$. **7**

(B) State and prove Weierstrass Approximation theorem. **7**

SECTION – II

9. Attempt any **FOUR** of the following in short : **8**

(1) Prove that X and ϕ are open sets.

(2) Define : Metric Space.

(3) If F is closed and K is compact then prove that $F \cap K$ is compact.

(4) Define : Connected set.

(5) Define : Uniform convergence.

(6) Prove by Taylor's series $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

(7) Is $f_n(x) = \frac{1}{1 + nx}$ $(x \geq 0)$ continuous ? Justify.

(8) Show that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$