Seat No. : _______________

JI-122

Janmuary-2021

B.Sc., Sem.-V

CC-303 :Mathematics
(Complex Variables and Fourier Series)

Time : 2 Hours] **[Max. Marks : 50**

Instructions : (1) Attempt any **Three** questions from Q. **1** to Q. **8**.

(2) Q. No. **9** is **Compulsory**.

(3) Notations are usual everywhere.

(4) Figures to the right indicate marks of the question/sub question.

1. (a) State and prove Triangle in equality in C. Hence deduce that

$\big||Z_1| - |Z_2|\big| \le |Z_1 - Z_2|,\ z_1, z_2 \in C.$ **7**

(b) In the system C of complex numbers show that

(i) $\bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}$

(ii) $\bar{z}_1 \cdot \bar{z}_2 = \overline{z_1 \cdot z_2}$

(iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2},\ z_2 \ne 0$

(iv) $|z_1 z_2| = |z_1|\,|z_2|$ **7**

2. (a) Define convergence of sequence and series in C. Suppose that $z_n = x_n + iy_n$; n = 1, 2, 3….. then prove that (z_n) converges to $z = x + iy$ if and only if (x_n) converges to x and (y_n) converges to y, where $z_n = x_n + iy_n$, n = 1, 2, 3… and $z = x + iy$. **7**

(b) Show that $\cos^2 z - \sin^2 z = \cos 2z$; $z = x + iy \in C$. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$. **7**

3. (a) Define : Harmonic function. If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then prove that component functions $u(x, y)$ and $v(x, y)$ are harmonic. Verify this result for the function $f(z) = z^2$. **7**

(b) Show that the function $f(z) = \dfrac{x^3 - y^3 + i(x^3 + y^3)}{x^2 + y^2}$; $(x, y) \neq (0, 0)$

and $f(z) = 0$; $(x, y) = (0, 0)$

is not analytic at $z = 0$ even if Cauchy-Riemann equations are satisfied at $z = 0$. **7**

4. (a) If the function $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic in the domain D then derive the equations $u_r = \frac{1}{r} v_\theta$ and $v_r = -\frac{1}{r} u_\theta$. Also, show that $v(r, \theta)$ satisfies the equation $r^2 v_{rr} + r v_r + v_{\theta\theta} = 0$. **7**

(b) If $u(x, y)$ and $v(x, y)$ are harmonic in D then prove that the function

$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is analytic. **7**

5. (a) Define conformal mapping and prove that an analytic function f(z) preserves conformality. **7**

(b) Find the image of the curve $|z - i| < 2$ under the mapping $w = \dfrac{iz + 1}{z + 2i}$. **7**

6. (a) Prove that the set of Bilinear Transformations form a non commutative group under the binary operation composition of two transformations, where, associativity is assumed. **7**

(b) Obtain the image of the curve $y = x - 1$ and $y = 0$ under the mapping $w = \frac{1}{z}$, $z \neq 0$. Also, check the conformality of this mapping at the point $z = -1$. **7**

7. (a) If the series $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ converges uniformly to f on $[-\pi, \pi]$, then prove that it is the Fourier series for f on $[-\pi, \pi]$. **7**

(b) Find a sine series for the function $f(x) = x$, for $0 < x < \frac{\pi}{2}$ and $f(x) = 0$ for $\frac{\pi}{2} < x < \pi$. **7**

8. (a) If $f(x)$ is Riemann integrable in $(-\pi, \pi)$, then the series $\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$ converges, where a_n and b_n are the Fourier coefficients of $f(x)$. **7**

(b) Find the Fourier series expansion of the function $f(x) = x^2$, in $0 < x < 2\pi$. **7**

9. Attempt any **four** of the followings in short : **8**

Take $z = x + iy$ a complex number to answer the short questions]

(a) State the Re(log z) and Im(log z).

(b) Is the curve $u = x^3 - 3xy^2$ Harmonic ? Justify.

(c) Which curve is represented by the expression $|3z - 2| = |z - i|$?

(d) Find the singular points of a mapping $f(z) = \dfrac{3z + 1}{(z - i)(z^2 - 8z + 15)}$.

(e) Is the function z^n Entire ? Explain.

(f) Obtain $\int_{-\pi}^{\pi} \cos nx \sin mx \, dx$ for all $m, n = 0, 1, 2, \ldots$.