3/22

1008E118

Candidate's Seat No : ______

M.Sc. Sem-3 Examination

501

Mathematics

August 2021

Time : 2-00 Hours] [Max. Marks : 50

Functional Analysis - I

Instructions:

1. All the questions in **Section-I** carry equal marks.
2. Attempt any **Three** questions in **Section-I**.
3. Questions in **Section-II** are **COMPULSORY**.

Section-I

1. (A) Let M and N be subspaces of vector space V, such that $V = M + N$. Show that $V = M \oplus N$ if and only if $M \cap N = \{0\}$. 7

   (B) Let N be a normed linear space. Prove the following: 7

   (i) For $x, y \in N$, $|\ \|x\| - \|y\|\ | \leq \|x - y\|$.

   (ii) Norm is a continuous function.

2. (A) Is the set $A = \{(x_1, x_2, x_3)/x_1 \text{ is an integer}\}$ a subspace of the real linear space $\mathbb{R}^3$? Justify your answer. 7

   (B) State and prove the closed graph theorem. 7

3. (A) Show that the vectors $(1, 0, 0), (0, 1, 0), (1, 1, 1)$ form a basis for $\mathbb{R}^3$. Show that if $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$, then $\{e_1 + e_2, e_1 + e_3, e_2 + e_3\}$ is also a basis. 7

   (B) State Parallelogram law. Is the parallelogram law true in $l_1^n (n > 1)$? Explain. 7

4. (A) Show that the set $\mathbb{R}^n$ of n-tuples $x = (x_1, \ldots, x_n)$ of real numbers is a Banach space under the norm 7

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}.$$

   (B) If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, prove that $(l_p^n)^* = l_q^n$. 7

5. (A) Let N be a non-zero normed linear space. If N is a Banach space, prove that $\{x : \|x\| = 1\}$ is complete. 7

   (B) For any non-empty subset S of a Hilbert space H, prove that $S^{\perp}$ is always a closed subspace of H. 7

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6. (a) If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M, prove that there exists a functional f_0 in N^* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$. 7

(b) State and prove Schwartz inequality. 7

7. (A) Let L be a non-zero finite-dimensional linear space of dimension n. Show that every set of $n+1$ vectors in L is linearly dependent. 7

(B) Give an example of a normed linear space which is not a Banach space. Justify your answer. 7

8. (A) Sketch the following sets: 7

(i) $S = \{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\|_2 = 1\}$.

(ii) $S = \{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\|_\infty = 1\}$.

(B) State the following theorems.(Do not prove) 7

(i) Open mapping theorem.

(ii) Uniform boundedness theorem.

Section-II

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(1) If M is a one-dimensional subspace of the real space $\mathbb{R}^3$, then

(A) M is a line through the origin.

(B) M is a plane through the origin.

(C) $M = \{0\}$.

(D) M is the entire space $\mathbb{R}^3$.

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(2) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be that linear transformation such that $T(1,1) = (1,-2)$ and $T(1,0) = (-4,1)$, then $T(5,-3)$ equals

(A) $(-35, 14)$ (B) $(-35, 6)$ (C) $(14, -35)$ (D) $(35, 3)$

(3) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping defined by $T(x,y,z) = (x - 3y - 2z, y - 4z, z)$, then $T^{-1}(x,y,z)$ equals

(A) $(x + 3y + 14z, y + 4z, z)$.

(B) $(x + 3y + 14z, y - 4z, z)$

(C) $(x + 3y - 14z, y - 4z, y - z)$

(D) $(x + 4z, 3x - 4y - z, x - y + z)$

(4) The inequality $\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \|y\|_q$ is called

(A) Cauchy's inequality

(B) Hölder's inequality.

(C) Minkowski's inequality.

(D) None of these.

(5) Which of the following spaces is not reflexive?

(A) l_p (B) l_p^n (C) c_0 (D) None of these

(6) Which of the following subspaces of normed linear space l_∞ is not closed?

(A) c (B) c_0 (C) c_{00} (D) None of these

(7) For $x \in X$, the norm in the inner product space X is

(A) $\|x\| = \langle x, x \rangle$

(B) $\|x\| = \sqrt{\langle x, x \rangle}$

(C) $\|x\| = \langle x, x \rangle^2$

(D) none of the above.

(8) If N is a finite-dimensional normed linear space of dimension n, then the dimension of conjugate space of N is

(A) equal to n

(B) less than n

(C) grater that n

(D) infinite