Seat No. : _______________

MJ-107

May-2022

B.Sc., Sem.-V

CC-303 : Mathematics

(Complex Variables and Fourier Series)

Time : 2 Hours] **[Max. Marks : 50**

Instructions : (1) Attempt any **three** questions from Q. 1 to Q. 8.
(2) Q. 9 is Compulsory.
(3) Notations are usual everywhere.
(4) Figures to the right indicate marks of the question/sub question.

1. (A) State the Triangle inequality in C and prove that **7**

(i) $\left||z_1| - |z_2|\right| \le |z_1 - z_2|$

(ii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

(B) In the system C of complex numbers obtain the fifth root of –1 and seventh root of 1. **7**

2. (A) Define convergence of series in C. Suppose that $z_n = a_n + ib_n; \quad n = 1,2,3...$ and S = A + iB then prove that $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ **7**

(B) Show that $ch^2z - sh^2z = 1$. Identify the curve |z + 1| = |z – 1| in C. Also, obtain Re $(|\sin z|^2)$ **7**

3. (A) Define : Analytic function. If a function *f* (z) = u (x, y) + iv (x, y) is analytic in a domain D, then derive the Cauchy-Riemann Partial differential equations. **7**

(B) Let $f\,(z) = \dfrac{(\bar{z})^2}{z}; z \neq 0$ and $f\,(z) = 0;\ z = 0$. Then show that *f* is not analytic at z = 0 even if Cauchy-Riemann equation are verified at z = 0. **7**

4. (A) Define : Harmonic function. If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in the domain D, then prove that the component functions $u(x, y)$ and $v(x, y)$ are harmonic. Find the harmonic conjugate of the function $x^2 - y^2$ and corresponding analytic function in terms of Z. **7**

(B) If $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic in the domain D, then prove that $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$. Verify the Cauchy-Riemann equations in polar form for the function $f(z) = \log z$. **7**

5. (A) Define conformal mapping and prove that an analytic function $f(z)$ preserves conformality. **7**

(B) Find the image of a strip $1 \leq y \leq 2$, x is a real number, under the mapping $w = \frac{1}{z}, z \neq 0$. **7**

6. (A) Find the image of the curve $|z - i| < 2$ under the Bilinear Transformation $w = \frac{iz+1}{z+2i}$. **7**

(B) Obtain the image of the curve $y = x - 1$ and $y = 0$ under the mapping $w = \frac{1}{z}, z \neq 0$

Also, examine the conformality of the given mapping at the point $z = -1$. **7**

7. (A) If the series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges uniformly to f on $[-\pi, \pi]$, then prove that it is the Fourier series for f on $[-\pi, \pi]$. **7**

(B) Find a sine series for the function $f(x) = x$, for $0 < x < \frac{\pi}{2}$

and $f(x) = 0$, for $\frac{\pi}{2} < x < \pi$. **7**

8. (A) If $f(x)$ is Riemann integrable in $(-\pi, \pi)$, then the series $\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ converges where a_n and b_n are the Fourier coefficients of $f(x)$. **7**

(B) Find the Fourier series expansion of the function $f(x) = x^2$, in $0 < x < 2\pi$. **7**

9. Attempt any **Four** of the followings in short : ($z = x + iy \in C$, every where) **8**

(i) State the $Re(e^z)$ and $Im(z^2)$.

(ii) Is the curve $u(x, y) = e^x \cos y$ Harmonic ? Justify.

(iii) Simplify the expression: $|z - 1| = |z - i|$?

(iv) Find the singular points of a mapping $f(z) = \dfrac{3z+1}{(z-i)(z^2-8z+15)}$.

(v) Is the function e^z Entire ? Explain.

(vi) Obtain $\int_{-\pi}^{\pi} \cos nx \sin mx \, dx$ for all m, n = 0, 1, 2, ...
